

HALL POLYNOMIALS FOR TAME TYPE

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ABSTRACT. In the present paper we prove that Hall polynomial exists for each triple of decomposition sequences which parameterize isomorphism classes of coherent sheaves of a domestic weighted projective line \mathbb{X} over finite fields. These polynomials are then used to define the generic Ringel–Hall algebra of \mathbb{X} as well as its Drinfeld double. Combining this construction with a result of Cramer, we show that Hall polynomials exist for tame quivers, which not only refines a result of Hubery, but also confirms a conjecture of Berenstein and Greenstein.

To the memory of Professor J. A. Green

1. INTRODUCTION

Inspired by the work of Steinitz [31] and Hall [13], Ringel [21, 22] introduced the Hall algebra $H(A)$ of a finite dimensional algebra A , whose structure constants are given by the so-called Hall numbers, and proved that if A is hereditary and representation finite, then $H(A)$ is isomorphic to the positive part of the corresponding quantized enveloping algebra. By introducing a bialgebra structure on $H(A)$, Green [11] then generalized Ringel’s work to arbitrary finite dimensional hereditary algebra A and showed that the composition subalgebra of $H(A)$ generalized by simple A -modules gives a realization of the positive part of the quantized enveloping algebra associated with A . The proof of the compatibility of multiplication and comultiplication on $H(A)$ is based on a marvelous formula arising from the homological properties of A -modules, called Green’s formula. We remark that Lusztig [16] has obtained a geometric construction of quantized enveloping algebras in terms of perverse sheaves on representation varieties of quivers.

In case A is representation finite and hereditary, Ringel [21] showed that the structure constants of $H(A)$ are actually integer polynomials in the cardinalities of finite fields. The proof is based on a basic property of the module category of A , namely, the directedness of its Auslander–Reiten quiver. These polynomials are called Hall polynomials as in the classical case; see [17]. Then one can define the generic Hall algebra $H_q(A)$ over the polynomial ring $\mathbb{Q}[q]$ and its degeneration $H_1(A)$ at $q = 1$. It was shown by Ringel [24] that $H_1(A)$ is isomorphic to the positive part of the universal enveloping algebra of the semisimple Lie algebra associated with A . Since then, much subsequent work was devoted to the study of Hall polynomials for various classes of algebras. Recently, Hubery [14] provided an elegant proof of the existence of Hall polynomials for all Dynkin and cyclic quivers by an inductive argument based on Green’s formula mentioned above. Moreover, he proved that Hall polynomials exist for all tame (affine) quivers with respect to the decomposition classes of Bongartz and Dudek [2].

2000 *Mathematics Subject Classification.* 17B37, 16G20.

Supported partially by the Natural Science Foundation of China.

Inspired by the work of Hubery [14], the main purpose of the present paper is to study Hall polynomials for coherent sheaves of a domestic weighted projective line \mathbb{X} over finite fields. The key idea is again the use of Green's formula. More precisely, we extend the notion of decomposition classes to that of decomposition sequences, which parameterize isoclasses (isomorphism classes) of coherent sheaves of \mathbb{X} over finite fields, and show that Hall polynomial exists for each triple of decomposition sequences. These polynomials are then applied to define an algebra $H_v(\mathbb{X})$ which is a free module over the Laurent polynomial ring $\mathbb{Q}[v, v^{-1}]$ with a basis all the decomposition sequences. By extending $H_v(\mathbb{X})$ via formally adding certain elements constructed in [4], we obtain the generic Ringel–Hall algebra $\mathcal{H}_v(\mathbb{X})$ of \mathbb{X} . By further introducing Green's pairing on $\mathcal{H}_v(\mathbb{X})$, we construct its Drinfeld double $D\mathcal{H}_v(\mathbb{X})$ over $\mathbb{Q}(v)$. Combining this construction with [6, Prop. 5], we show that Hall polynomials exist for decomposition sequences associated with a tame quiver. This result refines the main theorem of Hubery [14] and also confirms a conjecture of Berenstein and Greenstein [3, Conj. 3.4].

The paper is organized as follows. Section 2 gives a brief introduction on the category of coherent sheaves over a weighted projective line \mathbb{X} and recalls the definition of the Ringel–Hall algebra of \mathbb{X} over a finite field and the Green's formula as well. In Section 3 we define decomposition sequences for a domestic weighted projective line and give some preparatory results which are needed in Section 4 to prove the existence of Hall polynomials. Section 5 is devoted to defining the generic Hall algebra $\mathcal{H}_v(\mathbb{X})$ of \mathbb{X} as well as its Drinfeld double $D\mathcal{H}_v(\mathbb{X})$. In the final section, we show that Hall polynomials exist for tame quivers.

2. THE CATEGORY OF COHERENT SHEAVES OVER A WEIGHTED PROJECTIVE LINE

In this section we review the category of coherent sheaves over a weighted projective line and its basic properties, and we also introduce Hall algebras and Green's formula. For further fundamental concepts and facts on categories of coherent sheaves over weighted projective lines and on Hall algebras, we refer to [10, 5] and [26, 29, 7].

2.1. The category of coherent sheaves. Let k be an arbitrary field. A *weighted projective line* $\mathbb{X} = \mathbb{X}_k$ over k is specified by giving a *weight sequence* $\mathbf{p} = (p_1, p_2, \dots, p_t)$ of positive integers, and a collection $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_t)$ of distinct points in the projective line $\mathbb{P}^1(k)$ which can be normalized as $\lambda_1 = \infty, \lambda_2 = 0, \lambda_3 = 1$. More precisely, let $\mathbb{L} = \mathbb{L}(\mathbf{p})$ be the rank one abelian group with generators $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_t$ and the relations

$$p_1 \vec{x}_1 = p_2 \vec{x}_2 = \dots = p_t \vec{x}_t =: \vec{c},$$

where \vec{c} is called the *canonical element* of \mathbb{L} . Denote by S the commutative algebra

$$S = k[X_1, X_2, \dots, X_t] / \mathfrak{a} := k[x_1, x_2, \dots, x_t],$$

where $\mathfrak{a} = (f_3, \dots, f_t)$ is the ideal generated by $f_i = X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_1}, i = 3, \dots, t$. Put $I = \{1, 2, \dots, t\}$. Then S is \mathbb{L} -graded by setting

$$\deg(x_i) = \vec{x}_i \text{ for each } i \in I.$$

Moreover, each element $\vec{x} \in \mathbb{L}$ has the normal form $\vec{x} = \sum_{i \in I} l_i \vec{x}_i + l \vec{c}$ with $0 \leq l_i < p_i$ and $l \in \mathbb{Z}$. We denote by \mathbb{L}_+ the positive cone of \mathbb{L} which consists of those \vec{x} with $l \geq 0$. Finally, the weighted projective line associated with \mathbf{p} and $\boldsymbol{\lambda}$ is defined to be $\mathbb{X} = \text{Spec } \mathbb{L}S$.

According to [10], the set of nonzero prime homogeneous elements in S is partitioned into two sets: the exceptional primes x_1, \dots, x_t and the ordinary primes $f(x_1^{p_1}, x_2^{p_2})$, where $f(T_1, T_2)$ is a prime homogeneous polynomial in $k[T_1, T_2]$ which are distinct from T_1, T_2 and $T_2 - \lambda_i T_1$ for $i \in I$. The exceptional primes correspond to the points $\lambda_1, \dots, \lambda_t$, called exceptional points and denoted by x_1, \dots, x_t , respectively, while the ordinary primes correspond to the remaining closed points of $\mathbb{P}^1(k)$, called ordinary points. For convenience, we denote by \mathbb{H}_k the set of ordinary points. For each $z \in \mathbb{H}_k$, its degree $\deg(z)$ is defined to be the degree of the corresponding prime homogeneous polynomial.

The category of coherent sheaves on \mathbb{X} can be defined as the quotient of the category of finitely generated \mathbb{L} -graded S -modules over the Serre subcategory of finite length modules, that is,

$$\text{coh-}\mathbb{X} := \text{mod}^{\mathbb{L}}(S) / \text{mod}_0^{\mathbb{L}}(S).$$

The free module S gives the structure sheaf \mathcal{O} . Each line bundle is given by the grading shift $\mathcal{O}(\vec{x})$ for a uniquely determined element $\vec{x} \in \mathbb{L}$, and there is an isomorphism

$$\text{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) \cong S_{\vec{y}-\vec{x}}.$$

Moreover, $\text{coh-}\mathbb{X}$ is a hereditary abelian category with Serre duality of the form

$$D\text{Ext}^1(X, Y) \cong \text{Hom}(Y, X(\vec{\omega})),$$

where $D = \text{Hom}_k(-, k)$, and $\vec{\omega} := (t-2)\vec{c} - \sum_{i \in I} \vec{x}_i$ is called the *dualizing element* of \mathbb{L} . This implies the existence of almost split sequences in $\text{coh-}\mathbb{X}$ with the Auslander–Reiten translation τ given by the grading shift with $\vec{\omega}$.

Recall that $\text{coh-}\mathbb{X}$ admits a splitting torsion pair $(\text{coh}_0\text{-}\mathbb{X}, \text{vect-}\mathbb{X})$, where $\text{coh}_0\text{-}\mathbb{X}$ and $\text{vect-}\mathbb{X}$ are full subcategories of torsion sheaves and vector bundles, respectively. Moreover, $\text{coh}_0\text{-}\mathbb{X}$ decomposes as a direct product of orthogonal tubes

$$\text{coh}_0\text{-}\mathbb{X} = \prod_{z \in \mathbb{H}_k} \text{coh}_z\text{-}\mathbb{X} \times \prod_{i \in I} \text{coh}_i\text{-}\mathbb{X}$$

where each $\text{coh}_z\text{-}\mathbb{X}$ is a homogeneous tube, which is equivalent to the category of nilpotent representations of the Jordan quiver over the residue field k_z , while each $\text{coh}_i\text{-}\mathbb{X}$ is a non-homogeneous tube, which is equivalent to the category of nilpotent representations of the cyclic quiver with p_i vertices. By a classical result, the isoclasses (isomorphism classes) of objects in $\text{coh}_z\text{-}\mathbb{X}$ are indexed by partitions, while those in $\text{coh}_i\text{-}\mathbb{X}$ are indexed by multipartitions; see, for example, [25]. More precisely, for each $z \in \mathbb{H}_k$, $\text{coh}_z\text{-}\mathbb{X}$ admits a unique simple object S_z and, up to isomorphism, each object in $\text{coh}_z\text{-}\mathbb{X}$ has the form $S_k(\pi, z) = \bigoplus_{r=1}^s S_z[\pi_r]$, where $\pi = (\pi_1, \dots, \pi_s)$ is a partition and $S_z[\pi_r]$ is the unique indecomposable object of length π_r . While for each $i \in I$, there are p_i simple objects $S_{i,0}, \dots, S_{i,p_i-1}$ in $\text{coh}_i\text{-}\mathbb{X}$. For each $0 \leq j \leq p_i - 1$ and $l \geq 1$, let $S_{i,j}[l]$ denote the indecomposable object in $\text{coh}_i\text{-}\mathbb{X}$ of length l with top $S_{i,j}$.

It is known that the Grothendieck group $K_0(\mathbb{X})$ of $\text{coh-}\mathbb{X}$ is a free abelian group with a basis $\mathcal{O}(\vec{x})$ with $0 \leq \vec{x} \leq \vec{c}$, where we still write $X \in K_0(\mathbb{X})$ for the isoclass of an object $X \in \text{coh-}\mathbb{X}$. Let $p = \text{l.c.m.}(p_1, \dots, p_t)$ be the least common multiple of p_1, \dots, p_t and $\delta : \mathbb{L} \rightarrow \mathbb{Z}$ be the homomorphism defined by $\delta(\vec{x}_i) = \frac{p}{p_i}$. The *determinant* map is the group homomorphism

$$\det : K_0(\mathbb{X}) \longrightarrow \mathbb{L}, \quad \mathcal{O}(\vec{x}) \longmapsto \vec{x}.$$

The rank function on $K_0(\mathbb{X})$ is given by the rule $\text{rk}(\mathcal{O}(\vec{x})) = 1$ while the degree function is given by the rule $\deg(\mathcal{O}(\vec{x})) = \delta(\vec{x})$. For each non-zero object $X \in \text{coh-}\mathbb{X}$,

define the *slope* of X as $\mu(X) = \frac{\deg(X)}{\mathrm{rk}(X)}$. The Euler form on $K_0(\mathbb{X})$ is given by

$$\langle X, Y \rangle = \dim_k \mathrm{Hom}(X, Y) - \dim_k \mathrm{Ext}^1(X, Y).$$

for any $X, Y \in \mathrm{coh}\text{-}\mathbb{X}$. Its symmetrization is defined by

$$(X, Y) = \langle X, Y \rangle + \langle Y, X \rangle.$$

In the present paper we mainly focus on weighted projective lines \mathbb{X} of domestic type, i.e., $\delta(\vec{\omega}) < 0$. In this case, the Auslander–Reiten quiver $\Gamma(\mathrm{vect}\text{-}\mathbb{X})$ of $\mathrm{vect}\text{-}\mathbb{X}$ consists of a single standard component of the form $\mathbb{Z}\tilde{\Delta}$, where $\tilde{\Delta}$ is an extended Dynkin diagram associated with the weight sequence \mathbf{p} . Moreover, the full subcategory of indecomposable vector bundles on \mathbb{X} is equivalent to the mesh category of $\Gamma(\mathrm{vect}\text{-}\mathbb{X})$. Furthermore, for any two indecomposable objects $X, Y \in \mathrm{coh}\text{-}\mathbb{X}$, $\mathrm{Hom}(X, Y) \neq 0$ implies $\mu(X) \leq \mu(Y)$.

The following result will be needed later on.

Lemma 2.1. *Let E be an indecomposable vector bundle. Then there is an exact sequence $0 \rightarrow L \rightarrow E \rightarrow F \rightarrow 0$ in $\mathrm{vect}\text{-}\mathbb{X}$ such that L is a line bundle and $\mathrm{Ext}^1(F, L) \cong k$.*

Proof. Choose a line bundle L of maximal degree such that $\mathrm{Hom}(L, E) \neq 0$. Then we get an exact sequence

$$(2.1) \quad 0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0$$

in $\mathrm{coh}\text{-}\mathbb{X}$. We claim that F is a vector bundle. Otherwise, there exists a simple subsheaf S of F , which yields the following pullback commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & L' & \longrightarrow & S \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & F \longrightarrow 0. \end{array}$$

This gives a line bundle L' satisfying that $\mathrm{Hom}(L', E) \neq 0$ and $\deg L' > \deg L$, contradicting the choice of L .

Applying $\mathrm{Hom}(-, L)$ to the exact sequence (2.1) gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}(L, L) \longrightarrow \mathrm{Ext}^1(F, L) \longrightarrow \mathrm{Ext}^1(E, L).$$

Note that $\mathrm{Hom}(L, L) \cong k$ and $\mathrm{Ext}^1(E, L) \cong D\mathrm{Hom}(L(-\vec{\omega}), E) = 0$ since $\deg L(-\vec{\omega}) > \deg L$. Therefore, $\mathrm{Ext}^1(F, L) \cong k$. \square

2.2. The Hall algebra of coherent sheaves. Let k be a finite field. Given objects Z, X_1, \dots, X_t in $\mathrm{coh}\text{-}\mathbb{X}_k$, define F_{X_1, \dots, X_t}^Z to be the number of filtrations

$$Z = Z_0 \supseteq Z_1 \supseteq \dots \supseteq Z_{t-1} \supseteq Z_t = 0$$

such that $Z_{s-1}/Z_s \cong X_s$ for all $1 \leq s \leq t$, called the *Hall number* associated with Z, X_1, \dots, X_t .

For each object $X \in \mathrm{coh}\text{-}\mathbb{X}_k$, put $a_X = |\mathrm{Aut}(X)|$, the cardinality of the automorphism group $\mathrm{Aut}(X)$ of X . The following result is taken from [20, 18].

Lemma 2.2. *Let X, Y, Z be three objects in $\mathrm{coh}\text{-}\mathbb{X}_k$. Then*

$$F_{X, Y}^Z = \frac{|\mathrm{Ext}^1(X, Y)_Z|}{|\mathrm{Hom}(X, Y)|} \cdot \frac{a_Z}{a_X a_Y},$$

where $\text{Ext}^1(X, Y)_Z$ denotes the subset of $\text{Ext}^1(X, Y)$ consisting of equivalence classes of exact sequences in $\text{coh-}\mathbb{X}$ of the form $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$.

Now let k be a finite field with q elements and let v_q denote the square root \sqrt{q} of q . For each $M \in \text{coh-}\mathbb{X}_k$, let $[M]$ denote the isoclass of M . By definition, the Ringel–Hall algebra $H(\mathbb{X}_k)$ of the category of coherent sheaves on \mathbb{X}_k is the free module over the ring $\mathbb{Q}[v_q, v_q^{-1}]$ with basis $\{[M] \mid M \in \text{coh-}\mathbb{X}_k\}$, and the multiplication is given by

$$[M][N] = v_q^{\langle M, N \rangle} \sum_{[R], R \in \text{coh-}\mathbb{X}_k} F_{M, N}^R [R].$$

By a result of Green [11], $H(\mathbb{X}_k)$ is a bialgebra with comultiplication defined by

$$\Delta_k([R]) = \sum_{[M], [N]} v_q^{\langle M, N \rangle} F_{M, N}^R \frac{a_M a_N}{a_R} [M] \otimes [N].$$

In fact, the associativity of multiplication and the coassociativity of comultiplication follow from the identity

$$\sum_X F_{A, B}^X F_{X, C}^S = \sum_X F_{A, X}^S F_{B, C}^X,$$

where the sums on both sides are actually taken over isoclasses of objects in $\text{coh-}\mathbb{X}_k$, though we shall often use this more convenient notation. Furthermore, the compatibility of multiplication and comultiplication is encoded in following marvellous formula—the so-called Green’s formula—which plays a fundamental role in the study of Hall algebras.

Lemma 2.3 ([11]). *For each quadruple (M, N, X, Y) of objects in $\text{coh-}\mathbb{X}_k$, we have the equality*

$$(2.2) \quad \sum_E F_{M, N}^E F_{X, Y}^E / a_E = \sum_{A, B, C, D} q^{-\langle A, D \rangle} F_{A, B}^M F_{C, D}^N F_{A, C}^X F_{B, D}^Y \frac{a_A a_B a_C a_D}{a_M a_N a_X a_Y}.$$

Following an idea in [14, Sect. 3], if $\text{Ext}^1(X, Y) = 0$, then the left-hand side of (2.2) contains only one term

$$F_{M, N}^{X \oplus Y} F_{X, Y}^{X \oplus Y} / a_{X \oplus Y},$$

which, by Lemma 2.2, is equal to

$$F_{M, N}^{X \oplus Y} \frac{a_{X \oplus Y}}{|\text{Hom}(X, Y)| a_X a_Y} \cdot \frac{1}{a_{X \oplus Y}} = q^{-\langle X, Y \rangle} \frac{1}{a_X a_Y} F_{M, N}^{X \oplus Y}.$$

Thus, in this case, Green’s formula (2.2) is simplified to the form

$$(2.3) \quad F_{M, N}^{X \oplus Y} = \sum_{A, B, C, D} q^{\langle X, Y \rangle - \langle A, D \rangle} F_{A, B}^M F_{C, D}^N F_{A, C}^X F_{B, D}^Y \frac{a_A a_B a_C a_D}{a_M a_N}.$$

2.3. The elements $\Theta_{\vec{x}}$, T_r and Z_r in $H(\mathbb{X}_k)$. As above, let k be a finite field with q elements. In the following we recall from [4] the definition of some special elements in the Ringel–Hall algebra $H(\mathbb{X}_k)$ which will be needed later on.

By [4, 5.5], for each $\vec{x} \in \mathbb{L}_+$ with normal form $\vec{x} = \sum_{i \in I} l_i \vec{x}_i + l \vec{c}$, define the element $\Theta_{\vec{x}} = \Theta_{\vec{x}, q}$ via the formula

$$(2.4) \quad \Delta([\mathcal{O}]) = [\mathcal{O}] \otimes 1 + \sum_{\vec{x} \in \mathbb{L}_+} \Theta_{\vec{x}} \otimes [\mathcal{O}(-\vec{x})].$$

Then $\Theta_{\vec{x}}$ can be written as

$$(2.5) \quad \Theta_{\vec{x}} = v_q^{l+m} \sum_{z_j, n_j, m_i} \prod_j (1 - v_q^{-2 \deg(z_j)}) \times \prod_{i \in I, (m_i, l_i) \neq (0,0)} (1 - v_q^{-2}) \left[\bigoplus_j S_{z_j}[n_j] \oplus \bigoplus_{i \in I} S_{i,0}[m_i p_i + l_i] \right],$$

where $m = |\{i \mid l_i \neq 0\}|$, and the sum ranges over all tuples of distinct ordinary points z_j and nonnegative integers n_j, m_i satisfying $\sum_j n_j \deg(z_j) + \sum_{i \in I} m_i = l$.

However, the definition of the elements $T_r = T_{r,q}$ and $Z_r = Z_{r,q}$ in $H(\mathbb{X}_k)$ for all $r \geq 1$ are rather complicated, so we refer to [4, Sect. 6]. We emphasize that the Z_r commute with $[S]$ for all torsion sheaves S in $\text{coh-}\mathbb{X}_k$.

3. HALL POLYNOMIALS FOR A DOMESTIC WEIGHTED PROJECTIVE LINE

In this section, we introduce the notion of Hall polynomials and prove that Hall polynomials exist for a domestic weighted projective line.

As in the previous section, let $\mathbb{X} = \mathbb{X}_k$ be a domestic weighted projective line over a finite field k . By $\chi = \chi(\mathbb{X})$ we denote the set of isoclasses of objects in $\text{coh-}\mathbb{X}$ which clearly depends on the ground field k . Let χ_t and χ_f be the subsets of χ consisting of the isoclasses of torsion sheaves and vector bundles, respectively. Further, let χ_{nh} be the subset formed by the isoclasses of sheaves without homogeneous regular summands. In other words, χ_{nh} consists of isoclasses of sheaves whose indecomposable summands are either vector bundles or torsion sheaves lying in non-homogeneous tubes. Hence, the set χ_{nh} can be described combinatorially and is independent of k . Moreover, each sheaf in a homogeneous tube corresponding to a point in \mathbb{H}_k is determined by a partition. For each $\alpha \in \chi$, we fix a representative $S_k(\alpha)$ in the class α . Given $\alpha, \beta \in \chi$, we write $\alpha \oplus \beta$ for the isoclass of $S_k(\alpha) \oplus S_k(\beta)$. Thus, each $\alpha \in \chi$ can be uniquely decomposed as $\alpha = \alpha_t \oplus \alpha_f$ with $S_k(\alpha_t) \in \text{coh}_0\text{-}\mathbb{X}$ and $S_k(\alpha_f) \in \text{vect-}\mathbb{X}$.

3.1. Segre sequences and Hall polynomials. A *Segre sequence* is a sequence $\lambda = ((\lambda^{(1)}, d_1), (\lambda^{(2)}, d_2), \dots, (\lambda^{(r)}, d_r))$ of pairs $(\lambda^{(i)}, d_i)$, where $\lambda^{(i)}$ are partitions and d_i are positive integers with $d_1 \leq d_2 \leq \dots \leq d_r$. Such a sequence is said to be of type $\underline{d} = (d_1, d_2, \dots, d_r)$. If all $\lambda^{(i)}$ are the empty partition, then we simply write $\lambda = \emptyset$. A *decomposition sequence of type \underline{d}* is by definition a pair $\alpha = (\alpha, \lambda)$, where $\alpha \in \chi_{\text{nh}}$ and λ is a Segre sequence of type \underline{d} .

Remark 3.1. For a partition π and a positive integer d , by inserting the pair (π, d) to a Segre sequence $\lambda = ((\lambda^{(1)}, d_1), (\lambda^{(2)}, d_2), \dots, (\lambda^{(r)}, d_r))$ we mean the Segre sequence

$$\mu := ((\lambda^{(1)}, d_1), \dots, (\lambda^{(i)}, d_i), (\pi, d), (\lambda^{(i+1)}, d_{i+1}), \dots, (\lambda^{(r)}, d_r)),$$

where $1 \leq i \leq r$ satisfies $d_i \leq d < d_{i+1}$. In this case, we also say that λ is obtained from μ by removing the pair (π, d) . In particular, any finitely many Segre sequences can be converted to Segre sequences of same type via inserting and removing some pairs (\emptyset, d) . Finally, two decomposition sequences $\alpha = (\alpha, \lambda)$ and $\beta = (\beta, \mu)$ are identified if $\alpha = \beta$ and λ can be obtained from μ by inserting and removing some pairs (\emptyset, d) .

For a finite field k , denote by $\mathcal{X}_k(\underline{d})$ the set of sequences $\underline{z} = (z_1, \dots, z_r)$ of pairwise distinct points in \mathbb{H}_k with $\deg(z_i) = d_i$. Note that the cardinality of $\mathcal{X}_k(\underline{d})$ depends on the ground field k , and $\mathcal{X}_k(\underline{d})$ is possibly empty. However, $\mathcal{X}_k(\underline{d}) \neq \emptyset$ when $|k| \gg 0$.

For each Segre sequence λ of type \underline{d} and $\underline{z} = (z_1, \dots, z_r) \in \mathcal{X}_k(\underline{d})$, define

$$S_k(\lambda, \underline{z}) := \bigoplus_{i=1}^r S_k(\lambda^{(i)}, z_i) \in \text{coh-}\mathbb{X}_k,$$

where $S_k(\lambda^{(i)}, z_i) \in \text{coh}_{z_i}\text{-}\mathbb{X}_k$ is determined by the partition $\lambda^{(i)}$. Further, for each decomposition sequence $\alpha = (\alpha, \lambda)$ of type \underline{d} , define

$$S_k(\alpha, \underline{z}) := S_k(\alpha) \oplus S_k(\lambda, \underline{z}).$$

If $\lambda = \emptyset$, then $\alpha = (\alpha, \emptyset) \in \chi_{\text{nh}}$ and $S_k(\alpha) = S_k(\alpha)$. Clearly, $S_k(\alpha, \underline{z}) \in \text{vect-}\mathbb{X}_k$ if and only if $\alpha = \alpha_f \in \chi_f$ and $\lambda = \emptyset$. In this case, α is said to be of *torsion-free type* (we also simply write $\alpha \in \chi_f$), and we write $S_k(\alpha, \underline{z}) = S_k(\alpha) = S_k(\alpha)$. Furthermore, $S_k(\alpha, \underline{z}) \in \text{coh}_0\text{-}\mathbb{X}_k$ if and only if $\alpha \in \chi_{\text{nh}} \cap \chi_t$. In this case, α is said to be of *torsion type*.

Given $\alpha = (\alpha, \lambda)$ of type $\underline{d} = (d_1, \dots, d_r)$, if μ is obtained from λ by removing a pair (\emptyset, d_s) , where $1 \leq s \leq r$ and $d_s < d_{s+1}$, then μ is of type $\underline{d}' = (d_1, \dots, d_{s-1}, d_{s+1}, \dots, d_r)$ and, moreover, for each $\underline{z} = (z_1, \dots, z_r) \in \mathcal{X}_k(\underline{d})$,

$$S_k(\alpha, \underline{z}) \cong S_k((\alpha, \mu), \underline{z}'),$$

where $\underline{z}' = (z_1, \dots, z_{s-1}, z_{s+1}, \dots, z_r) \in \mathcal{X}_k(\underline{d}')$. Therefore, by setting $\beta = (\alpha, \mu)$, the two sets

$$\{S_k(\alpha, \underline{z}) \mid \underline{z} \in \mathcal{X}_k(\underline{d})\} \quad \text{and} \quad \{S_k(\beta, \underline{z}') \mid \underline{z}' \in \mathcal{X}_k(\underline{d}')\}$$

give rise to the same family of isoclasses of coherent sheaves in $\text{coh-}\mathbb{X}_k$.

By \mathcal{S} we denote the set of all decomposition sequences (up to the identification in Remark 3.1) and by \mathcal{S}_t (resp., \mathcal{S}_f) its subset of decomposition sequences of torsion type (resp., torsion-free type). Note that \mathcal{S}_f can be identified with χ_f which is a subset of χ_{nh} . Now for each $\alpha = (\alpha, \lambda) \in \mathcal{S}$, the decomposition $\alpha = \alpha_t \oplus \alpha_f$ gives two decomposition sequences

$$\alpha_t = (\alpha_t, \lambda) \in \mathcal{S}_t \quad \text{and} \quad \alpha_f = (\alpha_f, \emptyset) \in \mathcal{S}_f$$

such that for each $\underline{z} \in \mathcal{X}_k(\underline{d})$,

$$S_k(\alpha_t, \underline{z}) = S_k(\alpha, \underline{z})_t \quad \text{and} \quad S_k(\alpha_f, \underline{z}) = S_k(\alpha, \underline{z})_f = S_k(\alpha_f).$$

We simply write $\alpha = \alpha_t \oplus \alpha_f$.

Give $\alpha, \beta \in \mathcal{S}$ of type \underline{d} , it is easy to see from the definition that the value $\langle S_k(\alpha, \underline{z}), S_k(\beta, \underline{z}) \rangle$ is a constant for any field k with $|k| \gg 0$ and $\underline{z} \in \mathcal{X}_k(\underline{d})$. Thus, we simply put

$$\langle \alpha, \beta \rangle = \langle S_k(\alpha, \underline{z}), S_k(\beta, \underline{z}) \rangle \quad \text{and} \quad (\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle.$$

Definition 3.2. Given $\alpha, \beta, \gamma \in \mathcal{S}$ of type \underline{d} , if there exists a polynomial $\varphi_{\alpha, \beta}^\gamma \in \mathbb{Z}[T]$ such that for each finite field k of q elements with $q \gg 0$,

$$\varphi_{\alpha, \beta}^\gamma(q) = F_{S_k(\alpha, \underline{z}), S_k(\beta, \underline{z})}^{S_k(\gamma, \underline{z})} \quad \text{for all } \underline{z} \in \mathcal{X}_k(\underline{d}),$$

then we say that the Hall polynomial $\varphi_{\alpha, \beta}^\gamma$ exists for α, β and γ .

One of our main purposes in the present paper is to prove the following result.

Theorem 3.3. For arbitrary $\alpha, \beta, \gamma \in \mathcal{S}$ of type \underline{d} , the Hall polynomial $\varphi_{\alpha, \beta}^\gamma$ exists.

The theorem will be proved in the next section. In the following we present some preparatory results which will be needed for the proof of the main theorem.

Lemma 3.4 ([21]). *Let $\phi, \psi \in \mathbb{Z}[T]$ and assume ψ is monic. Then ψ divides ϕ if and only if the integer $\psi(q)$ divides the integer $\phi(q)$ for infinitely many $q \in \mathbb{Z}$.*

Lemma 3.5. *Given decomposition sequences α and β of type \underline{d} , there exists a monic integer polynomial $h_{\alpha, \beta} \in \mathbb{Z}[T]$ such that for any finite field k of q elements with $q \gg 0$,*

$$h_{\alpha, \beta}(q) = |\text{Hom}(S_k(\alpha, \underline{z}), S_k(\beta, \underline{z}))| \quad \text{for all } \underline{z} \in \mathcal{X}_k(\underline{d}).$$

Proof. Assume $\alpha = (\alpha, \lambda)$ and $\beta = (\beta, \mu)$ with $\alpha, \beta \in \chi_{\text{nh}}$ and λ, μ are Segre sequences. Since the Hom-functor commutes with direct sum, it suffices to prove the lemma when both $S_k(\alpha, \underline{z})$ and $S_k(\beta, \underline{z})$ are indecomposable. Thus, $\alpha = 0$ or $\lambda = \emptyset$, and $\beta = 0$ or $\mu = \emptyset$. We treat each of the four cases as follows:

- (i) *Case $\lambda = \emptyset$ and $\mu = \emptyset$.* Then the assertion follows from the fact that χ_{nh} can be described combinatorially.
- (ii) *Case $\alpha = 0$ and $\beta = 0$.* Then $S_k(\alpha, \underline{z})$ and $S_k(\beta, \underline{z})$ are Hom-orthogonal or belong to the same tube. This case follows from the representation theory of a cyclic quiver; see [25, 12].
- (iii) *Case $\lambda = \emptyset$ and $\beta = 0$.* Then $\text{Ext}^1(S_k(\alpha, \underline{z}), S_k(\beta, \underline{z})) = 0$. By Riemann–Roch formula in [10],

$$\text{rk } \alpha \deg \beta = \sum_{i=1}^p \langle S_k(\alpha, \underline{z}), \tau^i S_k(\beta, \underline{z}) \rangle = p \dim_k \text{Hom}(S_k(\alpha, \underline{z}), S_k(\beta, \underline{z})).$$

Thus,

$$\dim_k \text{Hom}(S_k(\alpha, \underline{z}), S_k(\beta, \underline{z})) = \frac{1}{p} \text{rk } \alpha \deg \beta,$$

which is an integer (since p divides $\deg \beta$) and independent of the field k , as desired.

- (iv) *Case $\alpha = 0$ and $\mu = \emptyset$.* This case can be proved by an argument similar to that in case (iii).

□

The following is an easy consequence of the lemma above.

Lemma 3.6. *Given a decomposition sequence α of type \underline{d} , there exists a monic integer polynomial $a_\alpha \in \mathbb{Z}[T]$ such that, for any finite field k of q elements with $q \gg 0$,*

$$a_\alpha(q) = |\text{Aut}(S_k(\alpha, \underline{z}))| \quad \text{for all } \underline{z} \in \mathcal{X}_k(\underline{d}).$$

Proposition 3.7. *If $\alpha, \beta, \gamma \in \mathcal{S}_f$, then the Hall polynomial $\varphi_{\alpha, \beta}^\gamma$ exists. If, moreover, $\text{Ext}^1(S_k(\alpha), S_k(\beta)) \cong k$, then $\varphi_{\alpha, \beta}^\gamma$ is monic.*

Proof. By the assumption, we can write $\alpha = (\alpha, \emptyset)$, $\beta = (\beta, \emptyset)$, and $\gamma = (\gamma, \emptyset)$ for $\alpha, \beta, \gamma \in \chi_f$.

Let k be a finite field and take a complete slice \mathcal{S} in $\text{vect-}\mathbb{X}_k$, which gives a tilting bundle T , such that every indecomposable direct summand of $S_k(\alpha), S_k(\beta)$ and $S_k(\gamma)$ is generated by T . For instance, the slice \mathcal{S} can be taken such that each indecomposable direct summand of $S_k(\alpha), S_k(\beta)$ and $S_k(\gamma)$ lies on the right hand side of the slice in the Auslander–Reiten quiver of $\text{vect-}\mathbb{X}_k$. It is well known that the endomorphism algebra $\Lambda := \text{End}(T)$ is tame hereditary and $\mathcal{F} := \text{Hom}(T, -)$ induces

an equivalence between certain exact full subcategories of $\text{coh-}\mathbb{X}_k$ and $\text{mod-}\Lambda$. In particular, the images of $S_k(\alpha)$, $S_k(\beta)$ and $S_k(\gamma)$ under \mathcal{F} belong to the preprojective component of $\text{mod-}\Lambda$. Then the existence of $\varphi_{\alpha,\beta}^\gamma$ follows from the fact that Hall polynomials exists for any three preprojective Λ -modules. (Note that the latter can be proved by using the arguments similar to those in [21, Th. 1].)

Now assume $\text{Ext}^1(S_k(\alpha), S_k(\beta)) \cong k$. Thus,

$$|\text{Ext}^1(S_k(\alpha), S_k(\beta))_{S_k(\gamma)}| = 0, 1, \text{ or } |k| - 1.$$

By Lemma 2.2, we have

$$F_{S_k(\alpha), S_k(\beta)}^{S_k(\gamma)} = \frac{|\text{Ext}^1(S_k(\alpha), S_k(\beta))_{S_k(\gamma)}|}{|\text{Hom}(S_k(\alpha), S_k(\beta))|} \frac{|\text{Aut}(S_k(\gamma))|}{|\text{Aut}(S_k(\alpha))||\text{Aut}(S_k(\beta))|}.$$

Then each term on the right hand side of the above equality is a monic integer polynomial in $|k|$. This implies that $\varphi_{\alpha,\beta}^\gamma$ is monic. \square

Proposition 3.8. *Let $\alpha = (\alpha, \lambda)$, $\beta = (\beta, \mu)$ and $\gamma = (\gamma, \nu)$ be decomposition sequences of type $\underline{d} = (d_1, \dots, d_r)$. If $\alpha, \beta, \gamma \in \mathcal{S}_t$, then the Hall polynomial $\varphi_{\alpha,\beta}^\gamma$ exists. Moreover, for fixed decomposition sequences α and β of torsion type, there are only finitely many decomposition sequences γ , and vice versa, such that $\varphi_{\alpha,\beta}^\gamma \neq 0$.*

Proof. Let k be a finite field. Since there are no nonzero homomorphisms between objects in distinct tubes of $\text{coh-}\mathbb{X}_k$, we have for each $\underline{z} = (z_1, \dots, z_r) \in \mathcal{X}_k(\underline{d})$,

$$\begin{aligned} F_{S_k(\alpha, \underline{z}), S_k(\beta, \underline{z})}^{S_k(\gamma, \underline{z})} &= F_{S_k(\alpha), S_k(\beta)}^{S_k(\gamma)} F_{S_k(\lambda, \underline{z}), S_k(\mu, \underline{z})}^{S_k(\nu, \underline{z})} \\ &= F_{S_k(\alpha), S_k(\beta)}^{S_k(\gamma)} \prod_{i=1}^r F_{S_k(\lambda^{(i)}, z_i), S_k(\mu^{(i)}, z_i)}^{S_k(\nu^{(i)}, z_i)}. \end{aligned}$$

By the existence of Hall polynomials for nilpotent representations of cyclic quivers (including the classical case; see [25, 12, 13, 17]), the Hall polynomials $\varphi_{\alpha,\beta}^\gamma(T)$ and $\varphi_{\lambda^{(i)}, \mu^{(i)}}^{\nu^{(i)}}(T)$ ($1 \leq i \leq r$) exist. Therefore, the polynomial

$$\varphi_{\alpha,\beta}^\gamma(T) \prod_{i=1}^r \varphi_{\lambda^{(i)}, \mu^{(i)}}^{\nu^{(i)}}(T^{d_i}) \in \mathbb{Z}[T]$$

is the required Hall polynomial $\varphi_{\alpha,\beta}^\gamma(T)$.

The second assertion follows from the properties of Hall polynomials for nilpotent representations of a cyclic quiver. \square

4. PROOF OF THEOREM 3.3

This section is devoted to proving Theorem 3.3. In the following we always assume that $\alpha = (\alpha, \lambda)$, $\beta = (\beta, \mu)$, and $\gamma = (\gamma, \nu)$ are decomposition sequences of type \underline{d} . Further, for a finite field k and $\underline{z} \in \mathcal{X}_k(\underline{d})$, we will simply put

$$M = M_k(\underline{z}) := S_k(\alpha, \underline{z}), \quad N = N_k(\underline{z}) := S_k(\beta, \underline{z}), \quad \text{and} \quad Z = Z_k(\underline{z}) := S_k(\gamma, \underline{z}).$$

4.1. Reduction 1. To prove Theorem 3.3, it suffices to prove the existence of Hall polynomials $\varphi_{\alpha,\beta}^\gamma$ for all $\alpha, \beta, \gamma \in \mathcal{S}$ with $\gamma \in \mathcal{S}_f = \chi_f$.

Proof. If γ is of torsion type and $F_{M_k(\underline{z}), N_k(\underline{z})}^{Z_k(\underline{z})} \neq 0$ for some finite field $k = \mathbb{F}_q$ and $\underline{z} \in \mathcal{X}_k(\underline{d})$, then both α and β are of torsion type. Thus, the existence of $\varphi_{\alpha,\beta}^\gamma$ follows from Proposition 3.8.

Now assume $Z_k(\underline{z}) = Z_k(\underline{z})_t \oplus Z_k(\underline{z})_f$ with $Z_k(\underline{z})_t \neq 0$ and $Z_k(\underline{z})_f \neq 0$ for some finite field $k = \mathbb{F}_q$ and $\underline{z} \in \mathcal{X}_k(\underline{d})$. Since $\text{Ext}^1(Z_k(\underline{z})_f, Z_k(\underline{z})_t) = 0$, applying the formula (2.3) to the quadruple $(M, N, Z_f, Z_t) = (M_k(\underline{z}), N_k(\underline{z}), Z_k(\underline{z})_f, Z_k(\underline{z})_t)$ gives the equality

$$F_{M,N}^Z = \sum_{A,B,C,D} q^{\langle Z_f, Z_t \rangle - \langle A, D \rangle} F_{A,B}^M F_{C,D}^N F_{A,C}^{Z_f} F_{B,D}^{Z_t} \frac{a_A a_B a_C a_D}{a_M a_N}.$$

On the one hand, $F_{B,D}^{Z_t} \neq 0$ implies that $B, D \in \text{coh}_0\text{-}\mathbb{X}_k$, i.e., $B = B_t$ and $D = D_t$, while $F_{A,C}^{Z_f} \neq 0$ implies $C \in \text{vect-}\mathbb{X}_k$, i.e., $C = C_f$. On the other hand, since $\text{Ext}^1(\text{vect-}\mathbb{X}_k, \text{coh}_0\text{-}\mathbb{X}_k) = 0$, we obtain that $D = N_t$ and $C = N_f$. Moreover, the associativity of Hall numbers implies that

$$F_{A,B_t}^M = \sum_E F_{A_f, A_t}^E F_{E, B_t}^M = \sum_E F_{A_f, E}^M F_{E, B_t}^E = \sum_{E_t} F_{A_f, E_t}^M F_{E_t, B_t}^{E_t} = \delta_{A_f, M_f} F_{A_t, B_t}^{M_t}.$$

Thus,

$$F_{M,N}^Z = \sum_{A, B \in \text{coh}_0\text{-}\mathbb{X}_k} q^{\langle Z_f, Z_t \rangle - \langle A \oplus M_f, N_t \rangle} F_{A,B}^{M_t} F_{A \oplus M_f, N_f}^{Z_f} F_{B, N_t}^{Z_t} \frac{a_{A \oplus M_f} a_B a_{N_f} a_{N_t}}{a_M a_N}.$$

Since $M_t = S_k(\alpha_t, \underline{z})$ with $\alpha_t = (\alpha_t, \lambda)$, $F_{A,B}^{M_t} \neq 0$ implies that

$$A = S_k(\xi, \underline{z}) \text{ and } B = S_k(\eta, \underline{z})$$

for $\xi, \eta \in \mathcal{S}$ of type \underline{d} . Moreover, by Proposition 3.8, there are only finitely many such pairs (ξ, η) . Hence, we conclude that

$$\begin{aligned} F_{M,N}^Z &= \sum_{\xi, \eta \in \mathcal{S}} q^{\langle Z_f, Z_t \rangle - \langle S_k(\xi, \underline{z}) \oplus M_f, N_t \rangle} F_{S_k(\xi, \underline{z}), S_k(\eta, \underline{z})}^{M_t} F_{S_k(\xi, \underline{z}) \oplus M_f, N_f}^{Z_f} F_{S_k(\eta, \underline{z}), N_t}^{Z_t} \\ &\quad \times \frac{a_{S_k(\xi, \underline{z}) \oplus M_f} a_{S_k(\eta, \underline{z})} a_{N_f} a_{N_t}}{a_M a_N}, \end{aligned}$$

where the sum, as indicated above, is essentially a finite sum. Applying Proposition 3.8 again shows that $F_{S_k(\xi, \underline{z}), S_k(\eta, \underline{z})}^{M_t}$ and $F_{S_k(\eta, \underline{z}), N_t}^{Z_t}$ are given by Hall polynomials $\varphi_{\xi, \eta}^{\alpha_t}$ and $\varphi_{\eta, \beta_t}^{\gamma_t}$, respectively. By Lemmas 3.4 and 3.6, the existence of Hall polynomial $\varphi_{\alpha, \beta}^{\gamma}$ follows from that of Hall polynomials $\varphi_{\xi \oplus \alpha_f, \beta_f}^{\gamma_f}$. \square

4.2. Reduction 2. To prove Theorem 3.3, it suffices to prove the existence of Hall polynomials $\varphi_{\alpha, \beta}^{\gamma}$ for all $\alpha \in \mathcal{S}_t$ and $\beta, \gamma \in \mathcal{S}_f$.

Proof. By Reduction 1, we can assume that $\gamma \in \mathcal{S}_f$ and, thus, $\beta \in \mathcal{S}_f$ since $\text{vect-}\mathbb{X}$ is closed under subobjects. Thus, $\gamma = (\gamma, \emptyset)$ and $\beta = (\beta, \emptyset)$ for some $\gamma, \beta \in \chi_f$. If $\alpha \in \mathcal{S}_f$, then the existence of $\varphi_{\alpha, \beta}^{\gamma}$ follows from Proposition 3.7.

Now assume $M = S_k(\alpha, \underline{z}) = M_t \oplus M_f$ with $M_t \neq 0$ and $M_f \neq 0$ for some finite field k and $\underline{z} \in \mathcal{X}_k(\underline{d})$. Associativity of Hall numbers implies that

$$F_{M,N}^Z = \sum_E F_{M_f, M_t}^E F_{E, N}^Z = \sum_E F_{M_f, E}^Z F_{M_t, N}^E,$$

where $N = S_k(\beta)$ and $Z = S_k(\gamma)$. The term $F_{M_f, E}^Z F_{M_t, N}^E \neq 0$ implies that $E \in \text{vect-}\mathbb{X}_k$ and there are embeddings $N \hookrightarrow E \hookrightarrow Z$. Thus, there are only finitely many

isoclasses of such E 's and $E = S_k(\theta)$ for some $\theta \in \mathcal{S}_f$. Hence,

$$F_{M,N}^Z = \sum_{\theta \in \chi_f} F_{M_f, S_k(\theta)}^Z F_{M_t, N}^{S_k(\theta)}.$$

By Proposition 3.7, $F_{M_f, S_k(\theta)}^Z$ is given by the Hall polynomial $\varphi_{\alpha_f, \theta}^\gamma$. Therefore, the existence of $\varphi_{\alpha, \beta}^\gamma$ follows from that of the $\varphi_{\alpha_t, \beta}^\theta$. \square

4.3. Reduction 3. Let t be an integer with $t \geq 2$. Suppose that Hall polynomials exist for all $\alpha \in \mathcal{S}_t$ and $\beta, \gamma \in \mathcal{S}_f$ with $\text{rk} \gamma < t$. Then the Hall polynomial $\varphi_{\alpha, \beta}^\gamma$ also exists for $\alpha \in \mathcal{S}_t$ and $\beta, \gamma \in \mathcal{S}_f$ with $\text{rk} \gamma = t$.

Proof. Take $\alpha \in \mathcal{S}_t$ and $\beta, \gamma \in \mathcal{S}_f$ with $\text{rk} \gamma = t$. Let $k = \mathbb{F}_q$ be a finite field. If $Z = S_k(\gamma)$ is decomposable, we can write $\gamma = \gamma_1 \oplus \gamma_2$ with $Z_1 = S_k(\gamma_1) \neq 0$, $Z_2 = S_k(\gamma_2) \neq 0$, and $\text{Ext}^1(Z_2, Z_1) = 0$. Applying (2.3) to the quadruple $(M = S_k(\alpha, \underline{z}), N = S_k(\beta), Z_2, Z_1)$, we obtain the equality

$$F_{M,N}^Z = \sum_{A,B,C,D} q^{\langle Z_2, Z_1 \rangle - \langle A, D \rangle} F_{A,B}^M F_{C,D}^N F_{A,C}^{Z_2} F_{B,D}^{Z_1} \frac{a_A a_B a_C a_D}{a_M a_N}.$$

On the one hand, $F_{A,C}^{Z_2} \neq 0$ implies that C is a subobject of Z_2 and

$$\deg C = \deg Z_2 - \deg A \geq \deg Z_2 - \deg M.$$

Thus, there are only finitely many such C 's and each of them has the form $C = S_k(\theta)$ for some $\theta \in \mathcal{S}_f$. Similarly, $F_{B,D}^{Z_1} \neq 0$ implies that $D = S_k(\sigma)$ for finitely many choices of $\sigma \in \mathcal{S}_f$. On the other hand, since $M = S_k(\alpha, \underline{z})$ is a torsion sheaf, we have by Proposition 3.8 that $F_{A,B}^M \neq 0$ implies that $A = S_k(\xi, \underline{z})$ and $B = S_k(\eta, \underline{z})$ for some $\xi, \eta \in \mathcal{S}_t$ of type \underline{d} and, moreover, there are only finitely many such pairs (ξ, η) . Therefore,

$$\begin{aligned} F_{M,N}^Z &= \sum_{\xi, \eta \in \mathcal{S}_t; \theta, \sigma \in \mathcal{S}_f} q^{\langle Z_2, Z_1 \rangle - \langle S_k(\xi, \underline{z}), S_k(\sigma) \rangle} F_{S_k(\xi, \underline{z}), S_k(\eta, \underline{z})}^M F_{S_k(\theta), S_k(\sigma)}^N F_{S_k(\xi, \underline{z}), S_k(\theta)}^{Z_2} \\ &\quad \times F_{S_k(\eta, \underline{z}), S_k(\sigma)}^{Z_1} \frac{a_{S_k(\xi, \underline{z})} a_{S_k(\eta, \underline{z})} a_{S_k(\theta)} a_{S_k(\sigma)}}{a_M a_N}. \end{aligned}$$

By Propositions 3.7 and 3.8, $F_{S_k(\delta), S_k(\sigma)}^N$ and $F_{S_k(\xi, \underline{z}), S_k(\eta, \underline{z})}^M$ are given by Hall polynomials $\varphi_{\delta, \sigma}^\gamma$ and $\varphi_{\xi, \eta}^\alpha$, respectively. Since $\text{rk} Z_1 < \text{rk} Z = t$ and $\text{rk} Z_2 < \text{rk} Z = t$, we have by the induction hypothesis that $F_{S_k(\xi, \underline{z}), S_k(\theta)}^{Z_2}$ and $F_{S_k(\eta, \underline{z}), S_k(\sigma)}^{Z_1}$ are given by Hall polynomials, too. Applying Lemmas 3.6 and 3.4 gives the existence of $\varphi_{\alpha, \beta}^\gamma$.

Now suppose that $Z = S_k(\gamma)$ is indecomposable. Since $\text{rk} Z = t \geq 2$, it follows from Lemma 2.1 that there is an exact sequence $0 \rightarrow L \rightarrow Z \rightarrow Z' \rightarrow 0$ with L a line bundle and Z' a vector bundle such that $\text{Ext}^1(Z', L) \cong k$. Applying (2.2) to the quadruple (M, N, Z', L) , we have

$$\sum_E F_{M,N}^E F_{Z',L}^E / a_E = \sum_{A,B,C,D} q^{-\langle A, D \rangle} F_{A,B}^M F_{C,D}^N F_{A,C}^{Z'} F_{B,D}^L \frac{a_A a_B a_C a_D}{a_M a_N a_{Z'} a_L}.$$

Since every extension of Z' by L is isomorphic either to Z or to $Z' \oplus L$, it follows that

$$\sum_E F_{M,N}^E F_{Z',L}^E / a_E = F_{M,N}^Z F_{Z',L}^Z / a_Z + F_{M,N}^{Z' \oplus L} F_{Z',L}^{Z' \oplus L} / a_{Z' \oplus L}.$$

Consequently, we obtain that

$$F_{M,N}^Z = \frac{a_Z}{F_{Z',L}^Z} \sum_{A,B,C,D} q^{-\langle A,D \rangle} F_{A,B}^M F_{C,D}^N F_{A,C}^{Z'} F_{B,D}^L \frac{a_A a_B a_C a_D}{a_M a_N a_{Z'} a_L} \\ - \frac{a_Z}{a_{Z' \oplus L} F_{Z',L}^Z} F_{M,N}^{Z' \oplus L} F_{Z',L}^{Z' \oplus L}.$$

By similar arguments as above, the sum on the right-hand side is finite and each Hall number occurring in the sum, as well as $F_{M,N}^{Z' \oplus L}$ (since $Z' \oplus L$ is decomposable), is given by an integer polynomial. By Lemma 2.2, it is direct to see that $F_{Z',L}^{Z' \oplus L}$ is given by an integer polynomial. Further, by Proposition 3.7, $F_{Z',L}^Z$ is given by a monic polynomial. Therefore, $\varphi_{\alpha,\beta}^\gamma$ exists by Lemma 3.4. \square

4.4. The proof of Theorem 3.3. Combining Reductions 1, 2 and 3, we are reduced to prove the existence of the Hall polynomials $\varphi_{\alpha,\beta}^\gamma$ for the case where $\alpha \in \mathcal{S}_t$, $\beta, \gamma \in \mathcal{S}_f$ with $\text{rk } \gamma = 1$.

Let k be a field and $\underline{z} = (z_1, \dots, z_r) \in \mathcal{X}_k(\underline{d})$. As above, put $Z = S_k(\gamma)$, $M = S_k(\alpha, \underline{z})$ and $N = S_k(\beta)$. By taking a grading shift, we may assume that $N = \mathcal{O}$ and $Z = \mathcal{O}(\vec{u})$ for some $\vec{u} \in \mathbb{L}_+$. We now use associativity together with an induction on the determinant $\det \vec{u}$ to prove the assertion.

If M supports at more than two distinct points, then M admits a non-trivial decomposition $M = M_1 \oplus M_2$ such that M_1 and M_2 have disjoint supports. It follows that

$$\text{Ext}^1(M_1, M_2) = 0 = \text{Ext}^1(M_2, M_1).$$

By associativity, we have

$$F_{M,\mathcal{O}}^{\mathcal{O}(\vec{u})} = \sum_E F_{M_2,M_1}^E F_{E,\mathcal{O}}^{\mathcal{O}(\vec{u})} = \sum_{\mathcal{O}(\vec{v})} F_{M_2,\mathcal{O}(\vec{v})}^{\mathcal{O}(\vec{u})} F_{M_1,\mathcal{O}}^{\mathcal{O}(\vec{v})} = \sum_{0 < \vec{v} < \vec{u}} F_{M_2(-\vec{v}),\mathcal{O}}^{\mathcal{O}(\vec{u}-\vec{v})} F_{M_1,\mathcal{O}}^{\mathcal{O}(\vec{v})}.$$

By induction on $\det \vec{u}$, all the terms on the right-hand side are given by Hall polynomials. This shows the existence of $\varphi_{\alpha,\beta}^\gamma$.

Now we assume that M supports at a single point. Then $F_{M,\mathcal{O}}^{\mathcal{O}(\vec{u})} \neq 0$ implies that there is a surjection $\mathcal{O}(\vec{u}) \twoheadrightarrow M$, which ensures that M is indecomposable; see, e.g., [29, Ex. 4.12]. If M has quasi-Loewy length $\ell \geq 2$, then there is an exact sequence $0 \rightarrow M' \rightarrow M \rightarrow S \rightarrow 0$, where S is quasi-simple. Associativity of Hall numbers implies that

$$F_{S,M'}^M F_{M,\mathcal{O}}^{\mathcal{O}(\vec{u})} + F_{S,M'}^{S \oplus M'} F_{S \oplus M',\mathcal{O}}^{\mathcal{O}(\vec{u})} = \sum_{\mathcal{O}(\vec{v})} F_{S,\mathcal{O}(\vec{v})}^{\mathcal{O}(\vec{u})} F_{M',\mathcal{O}}^{\mathcal{O}(\vec{v})} = \sum_{0 < \vec{v} < \vec{u}} F_{S(-\vec{v}),\mathcal{O}}^{\mathcal{O}(\vec{u}-\vec{v})} F_{M',\mathcal{O}}^{\mathcal{O}(\vec{v})}.$$

Clearly, $F_{S \oplus M',\mathcal{O}}^{\mathcal{O}(\vec{u})} = 0$ and $F_{S,M'}^M = 1$. Thus,

$$F_{M,\mathcal{O}}^{\mathcal{O}(\vec{u})} = \sum_{0 < \vec{v} < \vec{u}} F_{S(-\vec{v}),\mathcal{O}}^{\mathcal{O}(\vec{u}-\vec{v})} F_{M',\mathcal{O}}^{\mathcal{O}(\vec{v})}$$

is given by an integer polynomial by an inductive argument on $\det \vec{u}$.

Now assume M is a quasi-simple sheaf. If $M = S_{i,j}$ lies in a non-homogeneous tube, then $F_{S_{i,j},\mathcal{O}}^{\mathcal{O}(\vec{u})} = 1$ for $\vec{u} = \vec{x}_i$ and $j = 1$, and zero otherwise. If $M \in \text{coh}_z\text{-}\mathbb{X}_k$ for some $z \in \mathbb{H}_k$ with $r = \deg(z)$, then $F_{M,\mathcal{O}}^{\mathcal{O}(\vec{u})} = 1$ for $\vec{u} = r\vec{c}$ and it is zero otherwise. In both cases, the Hall polynomial $\varphi_{\alpha,\beta}^\gamma$ exists.

This finishes the proof of Theorem 3.3.

Let k be a finite field and $X \in \text{coh-}\mathbb{X}_k$. Then for each field extension $k \subseteq K$, $X^K := X \otimes_k K$ is an object in $\text{coh-}\mathbb{X}_K$. A finite field extension K of k is said to be *conservative* relative to X if for each indecomposable summand Y of X , Y^K is indecomposable in $\text{coh-}\mathbb{X}_K$. In general, given a finite set $\mathcal{X} = \{X_1, \dots, X_m\}$ of objects in $\text{coh-}\mathbb{X}_k$, a finite field extension K of k is said to be *conservative* relative to \mathcal{X} if K is conservative relative to each X_i for $1 \leq i \leq m$. Note that there always exist infinitely many conservative field extensions of k relative to \mathcal{X} .

Corollary 4.1. *Fix a finite field k and three objects M, N, Z in $\text{coh-}\mathbb{X}_k$. Then there exists a polynomial $\varphi_{M,N}^Z \in \mathbb{Z}[T]$ such that for each conservative field extension K of k relative to $\{M, N, Z\}$,*

$$\varphi_{M,N}^Z(|K|) = F_{M^K, N^K}^{Z^K}.$$

Proof. Choose decomposition sequences $\alpha = (\alpha, \lambda), \beta = (\beta, \mu), \gamma = (\gamma, \nu)$ of the same type, say of type $\underline{d} = (d_1, \dots, d_r)$, and $\underline{z} = (z_1, \dots, z_r) \in \mathcal{X}_k(\underline{d})$ such that

$$M = S_k(\alpha, \underline{z}), \quad N = S_k(\beta, \underline{z}), \quad \text{and} \quad Z = S_k(\gamma, \underline{z}).$$

Moreover, we can assume that for each $1 \leq i \leq r$, one of the partitions $\lambda^{(i)}, \mu^{(i)}, \nu^{(i)}$ is not the empty partition. Thus, if K is a conservative field extension of k relative to $\{M, N, Z\}$, then $\underline{z} \in \mathcal{X}_K(\underline{d})$ and

$$M^K = S_K(\alpha, \underline{z}), \quad N^K = S_K(\beta, \underline{z}), \quad \text{and} \quad Z^K = S_K(\gamma, \underline{z}).$$

By Theorem 3.3, $\varphi_{\alpha, \beta}^\gamma$ is the desired polynomial $\varphi_{M,N}^Z$. \square

5. GENERIC HALL ALGEBRA OF \mathbb{X} AND ITS DRINFELD DOUBLE

In this section, we define the (generic) Hall algebra of a domestic weighted projective line \mathbb{X} as well as its Drinfeld double by using the Hall polynomials given in the previous sections.

5.1. Generic Hall algebra. By 2.2, for each finite field k , we have the Ringel–Hall algebra $H(\mathbb{X}_k)$ of the category of coherent sheaves on \mathbb{X}_k defined over k .

Recall the set of decomposition classes \mathcal{S} over \mathbb{X} defined in Section 3 and the Hall polynomial $\varphi_{\alpha, \beta}^\gamma(T) \in \mathbb{Z}[T]$ for each triple $\alpha, \beta, \gamma \in \mathcal{S}$ of the same type. Let $\mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}]$ be the Laurent polynomial ring with indeterminate \mathbf{v} and put

$$H_{\mathbf{v}}(\mathbb{X}) := \bigoplus_{\alpha \in \mathcal{S}} \mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}] u_{\alpha},$$

that is, the free $\mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}]$ -module with basis $\{u_{\alpha} \mid \alpha \in \mathcal{S}\}$. For $\alpha, \beta \in \mathcal{S}$, define their multiplication by

$$u_{\alpha} u_{\beta} = \mathbf{v}^{\langle \alpha, \beta \rangle} \sum_{\gamma \in \mathcal{S}} \varphi_{\alpha, \beta}^\gamma(\mathbf{v}^2) u_{\gamma},$$

where α and β are thought of same type in the sense of Remark 3.1 and the sum is taken over all $\gamma \in \mathcal{S}$ of the same type. Note that for fixed $\alpha, \beta \in \mathcal{S}$, there are only finitely many γ satisfying $\varphi_{\alpha, \beta}^\gamma(T) \neq 0$. If $\alpha = (\alpha, \emptyset)$, we sometimes write $u_{\alpha} = u_{[S(\alpha)]}$ for computational purpose, e.g., $u_{[\emptyset]}$, $u_{[S_{i,j}]}$, etc.

Proposition 5.1. *The $\mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}]$ -module $H_{\mathbf{v}}(\mathbb{X})$ endowed with the multiplication defined above becomes an associative algebra with identity $1 = u_0$, where 0 denotes the decomposition class $(0, \emptyset)$.*

Proof. We need to show the associativity of the multiplication. Take arbitrary $\alpha, \beta, \gamma \in \mathcal{S}$. On the one hand, we have

$$\begin{aligned} (u_\alpha u_\beta) u_\gamma &= \mathbf{v}^{\langle \alpha, \beta \rangle} \sum_{\theta \in \mathcal{S}} \varphi_{\alpha, \beta}^\theta(\mathbf{v}^2) u_\theta u_\gamma = \mathbf{v}^{\langle \alpha, \beta \rangle} \sum_{\theta \in \mathcal{S}} \varphi_{\alpha, \beta}^\theta(\mathbf{v}^2) (\mathbf{v}^{\langle \theta, \gamma \rangle} \sum_{\delta \in \mathcal{S}} \varphi_{\theta, \gamma}^\delta(\mathbf{v}^2) u_\delta) \\ &= \mathbf{v}^{\langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle} \sum_{\delta \in \mathcal{S}} \left(\sum_{\theta \in \mathcal{S}} \varphi_{\alpha, \beta}^\theta(\mathbf{v}^2) \varphi_{\theta, \gamma}^\delta(\mathbf{v}^2) \right) u_\delta. \end{aligned}$$

On the other hand,

$$u_\alpha (u_\beta u_\gamma) = \mathbf{v}^{\langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle} \sum_{\delta \in \mathcal{S}} \left(\sum_{\theta \in \mathcal{S}} \varphi_{\alpha, \theta}^\delta(\mathbf{v}^2) \varphi_{\beta, \gamma}^\theta(\mathbf{v}^2) \right) u_\delta.$$

Thus, to prove the associativity, it suffices to show that

$$(5.1) \quad \sum_{\theta \in \mathcal{S}} \varphi_{\alpha, \beta}^\theta(T) \varphi_{\theta, \gamma}^\delta(T) = \sum_{\theta \in \mathcal{S}} \varphi_{\alpha, \theta}^\delta(T) \varphi_{\beta, \gamma}^\theta(T).$$

By the definition, for each finite field k with $q = |k| \gg 0$ and $\underline{z} \in \mathcal{X}_k(\underline{d})$,

$$\begin{aligned} \sum_{\theta \in \mathcal{S}} \varphi_{\alpha, \beta}^\theta(q) \varphi_{\theta, \gamma}^\delta(q) &= \sum_{\theta \in \mathcal{S}} F_{S_k(\alpha, \underline{z}), S_k(\beta, \underline{z})}^{S_k(\theta, \underline{z})} F_{S_k(\theta, \underline{z}), S_k(\gamma, \underline{z})}^{S_k(\delta, \underline{z})} \\ &= F_{S_k(\alpha, \underline{z}), S_k(\beta, \underline{z}), S_k(\gamma, \underline{z})}^{S_k(\delta, \underline{z})} = \sum_{\theta \in \mathcal{S}} \phi_{\alpha, \theta}^\delta(q) \phi_{\beta, \gamma}^\theta(q), \end{aligned}$$

where the second equality follows from the associativity of the Ringel–Hall algebra $H(\mathbb{X}_k)$. Hence, the left and right hand sides of (5.1) take the same values for prime power $q \gg 0$. In conclusion, (5.1) holds. \square

Form now onwards, let Ω denote the set of all prime powers ($\neq 1$). For each $q \in \Omega$, let \mathbb{F}_q denote the field with q elements and set $v_q = \sqrt{q}$. We will simply write $\mathbb{X}_q = \mathbb{X}_{\mathbb{F}_q}$, $S_q(\alpha, \underline{z}) = S_{\mathbb{F}_q}(\alpha, \underline{z})$, etc. Consider the infinite direct product

$$\prod_{q \in \Omega} H(\mathbb{X}_q)$$

which clearly carries a natural structure of an associative algebra whose multiplication is defined componentwise, that is, for any $(a_q)_q, (b_q)_q \in \prod_q H(\mathbb{X}_q)$,

$$(a_q)_q \cdot (b_q)_q = (a_q b_q)_q.$$

Let \mathcal{I} be the ideal of $\prod_q H(\mathbb{X}_q)$ generated by the elements $(a_q)_q$ with $a_q = 0$ for $q \gg 0$. Then the quotient

$$\widehat{H(\mathbb{X})} =: \prod_{q \in \Omega} H(\mathbb{X}_q) / \mathcal{I}$$

becomes an associative algebra, too. In other words, elements in $\widehat{H(\mathbb{X})}$ are equivalence classes under the equivalence relation \sim defined by

$$(a_q)_q \sim (b_q)_q \iff a_q = b_q \text{ for } q \gg 0.$$

It is clear that the element $\tilde{v} = (v_q)_q$ is invertible and does not satisfy any polynomial equation over \mathbb{Q} in $\widehat{H(\mathbb{X})}$. Thus, by identifying \tilde{v} with \mathbf{v} , $\widehat{H(\mathbb{X})}$ can be viewed as an algebra over the Laurent polynomial ring $\mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}]$.

In what follows, for each $q \in \Omega$, we fix a total ordering \preceq of all points in $\mathbb{H}_{\mathbb{F}_q}$ such that $x \preceq y$ implies $\deg(x) \leq \deg(y)$. The chain of points in $\mathbb{H}_{\mathbb{F}_q}$ of degree d is denoted by

$$y_{d,1} \prec y_{d,2} \prec \cdots \prec y_{d,\zeta_d},$$

where ζ_d is the number of points of degree d in $\mathbb{H}_{\mathbb{F}_q}$. For each given $\alpha \in \mathcal{S}$ of type $\underline{d} = (d_1, \dots, d_r)$, we fix an element $\underline{z} = \underline{z}_{\alpha,q} = (z_1, \dots, z_r) \in \mathcal{X}_{\mathbb{F}_q}(\underline{d})$ such that for each $d_1 \leq d \leq d_r$,

$$(z_i, \dots, z_j) = (y_{d,1}, \dots, y_{d,j-i+1}),$$

where $d_{i-1} < d = d_i = \dots = d_j < d_{j+1}$. In fact, \underline{z} is independent of α . In particular, if two decomposition sequences $\alpha = (\alpha, \lambda)$ and $\beta = (\beta, \mu)$ can be identified in the sense of Remark 3.1, then $S_q(\alpha, \underline{z}_{\alpha,q}) \cong S_q(\beta, \underline{z}_{\beta,q})$.

Proposition 5.2. *The assignment $u_\alpha \mapsto ([S_q(\alpha, \underline{z}_{\alpha,q})])_q, \alpha \in \mathcal{S}$, defines an embedding of $\mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}]$ -algebras*

$$\Phi : H_{\mathbf{v}}(\mathbb{X}) \longrightarrow \widehat{H(\mathbb{X})} = \prod_{q \in \Omega} H(\mathbb{X}_q)/\mathcal{I}.$$

Proof. The injectivity of Φ is obvious. We need to check that Φ is an algebra homomorphism. For any $\alpha, \beta \in \mathcal{S}$ (of same type), we have by the definition that

$$u_\alpha u_\beta = \mathbf{v}^{\langle \alpha, \beta \rangle} \sum_{\gamma \in \mathcal{S}} \varphi_{\alpha, \beta}^\gamma(\mathbf{v}^2) u_\gamma.$$

Furthermore, the q -component of $\Phi(u_\alpha)\Phi(u_\beta)$ is given by

$$[S_q(\alpha, \underline{z})][S_q(\beta, \underline{z})] = v_q^{\langle S_q(\alpha, \underline{z}), S_q(\beta, \underline{z}) \rangle} \sum_{S_q(\gamma, \underline{z})} F_{S_q(\alpha, \underline{z}), S_q(\beta, \underline{z})}^{S_q(\gamma, \underline{z})} [S_q(\gamma, \underline{z})],$$

where $\underline{z} = \underline{z}_{\alpha,q} = \underline{z}_{\beta,q} = \underline{z}_{\gamma,q}$, while the q -component of $\Phi(u_\alpha u_\beta)$ is given by

$$v_q^{\langle \alpha, \beta \rangle} \sum_{\gamma} \varphi_{\alpha, \beta}^\gamma(q) [S_q(\gamma, \underline{z})].$$

By Theorem 3.3, $\varphi_{\alpha, \beta}^\gamma(q) = F_{S_q(\alpha, \underline{z}), S_q(\beta, \underline{z})}^{S_q(\gamma, \underline{z})}$ for $q \gg 0$. Hence, $\Phi(u_\alpha u_\beta) = \Phi(u_\alpha)\Phi(u_\beta)$, as desired. \square

From now on, we will identify $H_{\mathbf{v}}(\mathbb{X})$ with the subalgebra $\Phi(H_{\mathbf{v}}(\mathbb{X}))$ of $\widehat{H(\mathbb{X})}$. Thus, we will use the notation u_α to denote its image $\Phi(u_\alpha)$ in $\widehat{H(\mathbb{X})}$.

Recall the elements $T_{r,q}$, $Z_{r,q}$ and $\Theta_{\vec{x},q}$ in $H(\mathbb{X}_q)$ introduced in §2.3 for all $r \geq 1$, $\vec{x} \in \mathbb{L}$, and $q \in \Omega$. Set

$$Z_r := (Z_{r,q})_q, \quad T_r := (T_{r,q})_q, \quad \Theta_{\vec{x}} := (\Theta_{\vec{x},q})_q \in \widehat{H(\mathbb{X})} = \prod_{q \in \Omega} H(\mathbb{X}_q)/\mathcal{I}.$$

Let $\mathcal{H}_{\mathbf{v}}(\mathbb{X})$ be the $\mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}]$ -subalgebra of $\widehat{H(\mathbb{X})}$ generated by $H_{\mathbf{v}}(\mathbb{X})$ and Z_r for $r \geq 1$ and put

$$\mathcal{H}_{\mathbf{v}}(\mathbb{X}) = \mathcal{H}_{\mathbf{v}}(\mathbb{X}) \otimes_{\mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}]} \mathbb{Q}(\mathbf{v}).$$

Both $\mathcal{H}_{\mathbf{v}}(\mathbb{X})$ and $\mathcal{H}_{\mathbf{v}}(\mathbb{X})$ are called the *generic Hall algebras* of \mathbb{X} .

Proposition 5.3. *For all $r \geq 1$ and $\vec{x} \in \mathbb{L}_+$, T_r and $\Theta_{\vec{x}}$ lie in $\mathcal{H}_{\mathbf{v}}(\mathbb{X})$. Moreover, $\mathcal{H}_{\mathbf{v}}(\mathbb{X})$ can be also generated by $H_{\mathbf{v}}(\mathbb{X})$ together with one of the following sets:*

$$(I) \{T_r \mid r \geq 1\}; \quad (II) \{\Theta_{r\vec{e}} \mid r \geq 1\}; \quad (III) \{\Theta_{\vec{x}} \mid \vec{x} \in \mathbb{L}_+\}.$$

Proof. By [4, Prop. 5.6], for every $q \in \Omega$ and $r > 0$, $T_{r,q} - Z_{r,q}$ belongs to the subalgebra of $H(\mathbb{X}_q)$ generated by $[S_{i,j}]$ for $i \in I$ and $0 \leq j \leq p_i - 1$. It follows that all $T_r - Z_r$ belong to the subalgebra of $\mathcal{H}_{\mathbf{v}}(\mathbb{X})$ generated by the $u_{[S_{i,j}]}$. Hence,

$$\langle H_{\mathbf{v}}(\mathbb{X}), T_r \mid r \geq 1 \rangle = \langle H_{\mathbf{v}}(\mathbb{X}), Z_r \mid r \geq 1 \rangle.$$

By [29, Ex. 4.12] and [4, Lem. 5.20], we get that

$$(5.2) \quad 1 + \sum_{r \geq 1} \Theta_{r\vec{c}} t^r = \exp((\mathbf{v} - \mathbf{v}^{-1}) \sum_{r \geq 1} T_r t^r).$$

This implies that

$$\langle H_{\mathbf{v}}(\mathbb{X}), T_r \mid r \geq 1 \rangle = \langle H_{\mathbf{v}}(\mathbb{X}), \Theta_{r\vec{c}} \mid r \geq 1 \rangle.$$

Moreover, by [4, Prop. 5.21], we have $\Theta_{\vec{x}} \in \langle H_{\mathbf{v}}(\mathbb{X}), \Theta_{r\vec{c}} \mid r \geq 1 \rangle$ for $\vec{x} \in \mathbb{L}_+$. This finishes the proof. \square

Let \mathcal{L} denote the set of infinite sequences of nonnegative integers $\underline{l} = (l_r)_{r \geq 1}$ satisfying $\sum_r l_r < \infty$. For each $\underline{l} \in \mathcal{L}$, set

$$Z_{\underline{l}} = \prod_{r \geq 1} Z_r^{l_r}, \quad T_{\underline{l}} = \prod_{r \geq 1} T_r^{l_r}, \quad \text{and} \quad \Theta_{\underline{l}} = \prod_{r \geq 1} \Theta_{r\vec{c}}^{l_r}.$$

Proposition 5.4. *Each of the following three sets*

$$\{u_{\alpha} Z_{\underline{l}} \mid \alpha \in \mathcal{S}, \underline{l} \in \mathcal{L}\}, \quad \{u_{\alpha} T_{\underline{l}} \mid \alpha \in \mathcal{S}, \underline{l} \in \mathcal{L}\}, \quad \text{and} \quad \{u_{\alpha} \Theta_{\underline{l}} \mid \alpha \in \mathcal{S}, \underline{l} \in \mathcal{L}\}$$

is a $\mathbb{Q}(\mathbf{v})$ -basis of $\mathcal{H}_{\mathbf{v}}(\mathbb{X})$.

Proof. We only prove that $\mathcal{Z} := \{u_{\alpha} Z_{\underline{l}} \mid \alpha \in \mathcal{S}, \underline{l} \in \mathcal{L}\}$ is a $\mathbb{Q}(\mathbf{v})$ -basis of $\mathcal{H}_{\mathbf{v}}(\mathbb{X})$. This together with the arguments in the proof of Proposition 5.3 implies that the other two sets are $\mathbb{Q}(\mathbf{v})$ -bases, too.

By the definition, $\mathcal{H}_{\mathbf{v}}(\mathbb{X})$ is spanned by \mathcal{Z} . It remains to show that \mathcal{Z} is a linearly independent set. For each $\alpha \in \mathcal{S}$ and $\underline{l} \in \mathcal{L}$, we have

$$u_{\alpha} Z_{\underline{l}} = u_{\alpha_f} u_{\alpha_t} Z_{\underline{l}},$$

where $\alpha = \alpha_f \oplus \alpha_t$ with $\alpha_f \in \mathcal{S}_f$ and $\alpha_t \in \mathcal{S}_t$. Then for each $q \in \mathfrak{Q}$, the q -th component of $u_{\alpha} Z_{\underline{l}}$ is

$$[S_q(\alpha_f)][S_q(\alpha_t, \underline{z}_{\alpha, q})] Z_{\underline{l}, q},$$

where $Z_{\underline{l}, q} = \prod_{r \geq 1} (Z_{r, q})^{l_r}$. Let $H^f(\mathbb{X}_q)$ (resp., $H^t(\mathbb{X}_q)$) be the $\mathbb{Q}[v_q, v_q^{-1}]$ -subalgebra of $H(\mathbb{X}_q)$ generated by $[S]$ with S all the torsion-free (resp., torsion) sheaves. Then the multiplication map

$$H^f(\mathbb{X}_q) \otimes_{\mathbb{Q}[v_q, v_q^{-1}]} H^t(\mathbb{X}_q) \xrightarrow{\text{mult}} H(\mathbb{X}_q)$$

is an isomorphism of $\mathbb{Q}[v_q, v_q^{-1}]$ -modules. Thus, it suffices to prove that the set

$$\{[S_q(\alpha, \underline{z}_{\alpha, q})] Z_{\underline{l}} \mid \alpha \in \mathcal{S}_t, \underline{l} \in \mathcal{L}\}$$

is linearly independent in $H(\mathbb{X}_q)$ for $q \gg 0$. By the construction of $Z_{r, q}$ given in [29], this is reduced to prove that for each fixed $\underline{l} \in \mathcal{L}$, the set $\{[S_q(\alpha, \underline{z}_{\alpha, q})] Z_{\underline{l}} \mid \alpha \in \mathcal{S}_t\}$ is linearly independent.

By [27, 15] and [4, Sect. 5], the elements $Z_{r, q}$ are central in $H^t(\mathbb{X}_q)$, and for each $\underline{l} \in \mathcal{L}$, multiplication by $Z_{\underline{l}, q}$

$$H^t(\mathbb{X}_q) \longrightarrow H^t(\mathbb{X}_q), \quad a \longmapsto a Z_{\underline{l}, q}$$

is injective. Therefore, for $q \gg 0$, the set

$$\{[S_q(\alpha, \underline{z})] Z_{\underline{l}, q} \mid \alpha \in \mathcal{S}_t\}$$

is linearly independent. We conclude that

$$\mathcal{Z} = \{u_{\alpha} Z_{\underline{l}} \mid \alpha \in \mathcal{S}, \underline{l} \in \mathcal{L}\}$$

is a linearly independent set, as desired. \square

Proposition 5.5. *The following relations hold in $\mathcal{H}_v(\mathbb{X})$:*

- (1) $[Z_r, Z_s] = 0$;
- (2) $[Z_r, u_{\alpha_t}] = 0$;
- (3) $[Z_r, u_{[\mathcal{O}(\vec{x})]}] = \gamma_r u_{[\mathcal{O}(\vec{x}+r\vec{c})]}$ for some $\gamma_r \in \mathbb{Q}(v)$.
- (4) $[T_r, u_{[\mathcal{O}(\vec{x})]}] = \frac{[2r]}{r} u_{[\mathcal{O}(\vec{x}+r\vec{c})]}$, where $[2r] = (v^{2r} - v^{-2r})/(v - v^{-1})$.
- (5) $[\Theta_{r\vec{c}}, u_{[\mathcal{O}(n\vec{c})]}] = (1 - v^{-4}) \sum_{1 \leq i \leq r} v^{2i} u_{[\mathcal{O}((n+i)\vec{c})]} \Theta_{(r-i)\vec{c}}$.

Proof. The relations (1)–(4) hold since they hold in each q -component for $q \in \mathfrak{Q}$; see [4].

We now prove (5). For each finite field $k = \mathbb{F}_q$, the well-known embedding $\text{coh-}\mathbb{P}^1(k) \rightarrow \text{coh-}\mathbb{X}_k$ induces an algebra embedding between their Ringel–Hall algebras which takes $\Theta_{r,q} \mapsto \Theta_{r\vec{c},q}$ and $[\mathcal{O}(n)] \mapsto [\mathcal{O}(n\vec{c})]$; see [4, Lem. 5.20]. Hence, it suffices to show that the equality

$$(5.3) \quad [\Theta_{r,q}, [\mathcal{O}(n)]] = (1 - v^{-4}) \sum_{1 \leq i \leq r} v^{2i} [\mathcal{O}((n+i))] \Theta_{r-i,q}$$

holds in the Ringel–Hall algebra of $\mathbb{P}^1(k)$, where $v = \sqrt{q}$. Consider the generating functions

$$\Theta(s) = 1 + \sum_{i \geq 1} \Theta_{i,q} s^i, \quad \mathbb{T}(s) = 1 + \sum_{i \geq 1} \frac{T_i}{[i]_v} s^i, \quad \text{and} \quad \mathbb{O}(t) = 1 + \sum_{i \geq 1} [\mathcal{O}(i)] t^i,$$

where $[i]_v = (v^i - v^{-i})/(v - v^{-1})$. By the proof of [29, Ex. 4.12],

$$\Theta(s) = \exp(\mathbb{T}(vs) - \mathbb{T}(v^{-1}s)) \quad \text{and} \quad [\mathbb{T}(s), \mathbb{O}(t)] = -\mathbb{O}(t) \log(1 - \frac{s}{vt})(1 - \frac{vs}{t}).$$

Hence,

$$[\mathbb{T}(vs) - \mathbb{T}(v^{-1}s), \mathbb{O}(t)] = \mathbb{O}(t) \log \frac{1 - \frac{s}{v^2 t}}{1 - \frac{v^2 s}{t}}.$$

This implies that

$$\Theta(s) \mathbb{O}(t) = \mathbb{O}(t) \Theta(s) \frac{1 - \frac{s}{v^2 t}}{1 - \frac{v^2 s}{t}}.$$

Then (5.3) follows from the equality

$$\frac{1 - \frac{s}{v^2 t}}{1 - \frac{v^2 s}{t}} = 1 + (1 - v^{-4}) \sum_{i \geq 1} v^{2i} \frac{s^i}{t^i}.$$

□

5.2. Extended generic Hall algebra. For each prime power $q \in \mathfrak{Q}$, let $K_0(\mathbb{X}_q)$ be the Grothendieck group of $\text{coh-}\mathbb{X}_q$. It is well known that $K_0(\mathbb{X}_q)$ is independent of q . Thus, we simply write $K_0(\mathbb{X})$ for $K_0(\mathbb{X}_q)$. Finally, set

$$\delta := [\mathcal{O}(\vec{c})] - [\mathcal{O}] \in K_0(\mathbb{X}).$$

Furthermore, we denote by $\mathcal{K} = \mathbb{Q}[v, v^{-1}][K_0(\mathbb{X})]$ the group algebra of $K_0(\mathbb{X})$. To avoid possible confusion, we write $K_{[M]}$ instead of $[M]$ for each $M \in \text{coh-}\mathbb{X}$. Furthermore, for each decomposition sequence α in \mathcal{S} , we simply set $K_\alpha = K_{[S_q(\alpha, \mathbb{Z})]}$ in \mathcal{K} for $q \gg 0$. We equip the $\mathbb{Q}[v, v^{-1}]$ -module $\tilde{\mathcal{H}}_v(\mathbb{X}) := \mathcal{H}_v(\mathbb{X}) \otimes_{\mathbb{Q}[v, v^{-1}]} \mathcal{K}$ with an algebra structure (containing $\mathcal{H}_v(\mathbb{X})$ and \mathcal{K} as subalgebras) by imposing the relations

$$K_a u_\alpha K_a^{-1} = v^{(\mathbf{a}, \alpha)} u_\alpha, \quad \forall \alpha \in \mathcal{S}, a \in K_0(\mathbb{X}).$$

For each $q \in \Omega$, denote by $\tilde{H}(\mathbb{X}_q)$ the extended version of $H(\mathbb{X}_q)$, that is,

$$\tilde{H}(\mathbb{X}_q) = H(\mathbb{X}_q) \otimes_{\mathbb{Q}[v_q, v_q^{-1}]} \mathbb{Q}[v_q, v_q^{-1}][K_0(\mathbb{X})].$$

Then the embedding of $\mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}]$ -algebras

$$\Phi : H_{\mathbf{v}}(\mathbb{X}) \longrightarrow \widehat{H(\mathbb{X})} = \prod_{q \in \Omega} H(\mathbb{X}_q)/\mathcal{I}$$

extends to an embedding

$$\tilde{\Phi} : \tilde{H}_{\mathbf{v}}(\mathbb{X}) \longrightarrow \prod_{q \in \Omega} \tilde{H}(\mathbb{X}_q)/\tilde{\mathcal{I}}, \quad u_{\alpha} K_{\mathbf{a}} \longmapsto ([S_q(\alpha, \underline{z})] K_{\mathbf{a}})_q,$$

where $\tilde{\mathcal{I}}$ denotes the ideal of $\prod_q \tilde{H}(\mathbb{X}_q)$ generated by the elements $(a_q)_q$ with $a_q = 0$ for $q \gg 0$. We view $\tilde{H}_{\mathbf{v}}(\mathbb{X})$ as a subalgebra of $\prod_q \tilde{H}(\mathbb{X}_q)/\tilde{\mathcal{I}}$ and denote by $\tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X})$ the $\mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}]$ -subalgebra generated by $\tilde{H}_{\mathbf{v}}(\mathbb{X})$ together with Z_r for $r \geq 1$. Finally, we set

$$\tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X}) = \tilde{H}_{\mathbf{v}}(\mathbb{X}) \otimes_{\mathbb{Q}[\mathbf{v}, \mathbf{v}^{-1}]} \mathbb{Q}(\mathbf{v}) = \mathcal{H}_{\mathbf{v}}(\mathbb{X}) \otimes_{\mathbb{Q}(\mathbf{v})} \mathbb{Q}(\mathbf{v})[K_0(\mathbb{X})].$$

Both $\tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X})$ and $\mathcal{H}_{\mathbf{v}}(\mathbb{X})$ are called the *extended generic Hall algebras* of \mathbb{X} .

The topological comultiplications Δ_q on the Hall algebra $\tilde{H}(\mathbb{X}_q)$ for all $q \in \Omega$ induce a topological comultiplication Δ on $\prod_q \tilde{H}(\mathbb{X}_q)$ by $\Delta((a_q)_q) = (\Delta_q(a_q))_q$, where $a_q \in \tilde{H}(\mathbb{X}_q)$. It is easy to see that $\tilde{\mathcal{I}}$ is a coideal. Hence, it induces a topological comultiplication on the quotient $\prod_q \tilde{H}(\mathbb{X}_q)/\tilde{\mathcal{I}}$, which is still denoted by Δ .

Proposition 5.6. *We have the following formulas in $\tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X})$:*

- (1) $\Delta(u_{\gamma} K_{\mathbf{a}}) = \sum_{\alpha, \beta} \mathbf{v}^{\langle \alpha, \beta \rangle} \varphi_{\alpha, \beta}^{\gamma} \frac{a_{\alpha} a_{\beta}}{a_{\gamma}} u_{\alpha} K_{\beta + \mathbf{a}} \otimes u_{\beta} K_{\mathbf{a}}$ for $\gamma \in \chi_t$ and $\mathbf{a} \in K_0(\mathbb{X})$;
- (2) $\Delta(u_{[\mathcal{O}]}) = u_{[\mathcal{O}]} \otimes 1 + \sum_{\vec{x} \in \mathbb{L}_+} \Theta_{\vec{x}} K_{[\mathcal{O}(-\vec{x})]} \otimes u_{[\mathcal{O}(-\vec{x})]}$;
- (3) $\Delta(Z_r) = Z_r \otimes 1 + K_{r\delta} \otimes Z_r$ for $r \geq 1$.

Proof. By definition, it suffices to show that each formula holds for any finite field k with $q := |k| \gg 0$. Thus, the formulas (2) and (3) follow directly from [4].

We now prove (1). Since for $\underline{z} = \underline{z}_{\gamma, q}$, both subsheaves and quotient sheaves of $S_q(\gamma, \underline{z})$ are again torsion sheaves with supports in \underline{z} , we have

$$\begin{aligned} & \Delta_q([S_q(\gamma, \underline{z})] K_{\mathbf{a}}) \\ &= \sum_{\alpha, \beta} v_q^{\langle \alpha, \beta \rangle} F_{S_q(\alpha, \underline{z}), S_q(\beta, \underline{z})}^{S_q(\gamma, \underline{z})} \frac{a_{S_q(\alpha, \underline{z})} a_{S_q(\beta, \underline{z})}}{a_{S_q(\gamma, \underline{z})}} [S_q(\alpha, \underline{z})] K_{\beta + \mathbf{a}} \otimes [S_q(\beta, \underline{z})] K_{\mathbf{a}} \\ &= \sum_{\alpha, \beta} v_q^{\langle \alpha, \beta \rangle} \varphi_{\alpha, \beta}^{\gamma}(q) \frac{a_{\alpha}(q) a_{\beta}(q)}{a_{\gamma}(q)} [S_q(\alpha, \underline{z})] K_{\beta + \mathbf{a}} \otimes [S_q(\beta, \underline{z})] K_{\mathbf{a}}. \end{aligned}$$

□

Proposition 5.7. *The comultiplication Δ induces a topological bialgebra structure on the extended generic Hall algebra $\tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X})$.*

Proof. By Proposition 5.6, for $r \geq 1$ and $\alpha \in \chi_t$, $\Delta(Z_r)$ and $\Delta(u_{\alpha} K_{\mathbf{a}})$ belong to the completion $\tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X}) \hat{\otimes} \tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X})$. It suffices to show that $\Delta(u_{\alpha}) \in \tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X}) \hat{\otimes} \tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X})$ for $\alpha \in \chi_f$. Note that for each $q \in \Omega$, the category $\text{vect-}\mathbb{X}_q$ can be generated by line bundles. It is reduced to prove that $\Delta(u_{[\mathcal{O}(\vec{x})]})$ belongs to $\tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X}) \hat{\otimes} \tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X})$. Since

$\tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X})$ is closed under grading shift by \vec{x} ; see [4, Cor. 5.18], the assertion follows from Propositions 5.3, Proposition 5.6(2), and the fact that $\Theta_{\vec{x}} \in \tilde{\mathcal{H}}_{\mathbf{v}}(\mathbb{X})$ for all $\vec{x} \in \mathbb{L}_+$. \square

5.3. Drinfeld double of extended generic Hall algebra. By [11], for each $q \in \mathfrak{Q}$, there is a paring (called the Green's pairing)

$$\{-, -\}_q : \tilde{H}(\mathbb{X}_q) \times \tilde{H}(\mathbb{X}_q) \longrightarrow \mathbb{Q}[v_q, v_q^{-1}]$$

on $\tilde{H}(\mathbb{X}_q)$ defined by

$$\{[S]K_{\mathbf{a}}, [S']K_{\mathbf{b}}\}_q = v_q^{(\mathbf{a}, \mathbf{b})} \frac{\delta_{S, S'}}{a_S} \quad \forall S, S' \in \text{coh-}\mathbb{X}_q, \mathbf{a}, \mathbf{b} \in K_0(\mathbb{X}).$$

Moreover, this pairing is non-degenerate and symmetric, and satisfies

$$\{ab, c\}_q = \{a \otimes b, \Delta_q(c)\}_q \quad \forall a, b, c \in \tilde{H}(\mathbb{X}_q).$$

They give rise to a pairing

$$\{-, -\} : \prod_q \tilde{H}(\mathbb{X}_q) \times \prod_q \tilde{H}(\mathbb{X}_q) \longrightarrow \prod_q \mathbb{Q}[v_q, v_q^{-1}]$$

defined by

$$\{(a_q)_q, (b_q)_q\} = (\{a_q, b_q\}_q)_q.$$

It is easy to see that $\bigoplus_q \mathbb{Q}[v_q, v_q^{-1}]$ is an ideal of $\prod_q \mathbb{Q}[v_q, v_q^{-1}]$, and we denote the quotient ring by $\tilde{\mathbb{Q}}$. Finally, we obtain a pairing

$$(5.4) \quad \{-, -\} : \left(\prod_q \tilde{H}(\mathbb{X}_q) / \tilde{\mathcal{I}} \right) \otimes \left(\prod_q \tilde{H}(\mathbb{X}_q) / \tilde{\mathcal{I}} \right) \longrightarrow \tilde{\mathbb{Q}}$$

satisfying that

$$\{(a_q)_q, (b_q)_q, (c_q)_q\} = \{(a_q)_q \otimes (b_q)_q, \Delta((c_q)_q)\}_q, \quad \forall (a_q)_q, (b_q)_q, (c_q)_q \in \prod_q \tilde{H}(\mathbb{X}_q) / \tilde{\mathcal{I}}.$$

Now we recall some notations from [4]. For each $1 \leq i \leq t$ and $r > 0$, define

$$\text{def}_{x_i} = \frac{[r]^2}{r} \left(\frac{1}{\mathbf{v}^{2rp_i} - 1} - \frac{1}{\mathbf{v}^{2r} - 1} \right)$$

which is the defect of the exceptional point x_i when specializing \mathbf{v} to v_q , where $[r] = (\mathbf{v}^r - \mathbf{v}^{-r}) / (\mathbf{v} - \mathbf{v}^{-1})$.

Proposition 5.8. *We have the following equalities:*

- (1) $\{Z_r, Z_s\} = \delta_{r,s} \left(\frac{1}{\mathbf{v} - \mathbf{v}^{-1}} \frac{[2r]}{r} + \sum_{i \in I} \text{def}_{x_i} \right), \quad \forall r, s \geq 1;$
- (2) $\{u_{\alpha} K_{\mathbf{a}}, u_{\beta} K_{\mathbf{b}}\} = \mathbf{v}^{(\mathbf{a}, \mathbf{b})} \frac{\delta_{\alpha, \beta}}{a_{\alpha}}, \quad \forall \alpha, \beta \in \mathcal{S}, \mathbf{a}, \mathbf{b} \in K_0(\mathbb{X});$
- (3) For $\vec{x} = \sum_{i \in I} l_i \vec{x}_i + l \vec{c} \in \mathbb{L}_+$ and $\alpha = (\alpha, \lambda) \in \mathcal{S}$ with $\lambda = ((\lambda^{(1)}, d_1), \dots, (\lambda^{(r)}, d_r))$, $\{u_{\alpha}, \Theta_{\vec{x}}\} = 0$ or equals to

$$\frac{\mathbf{v}^{(\alpha, \alpha) + l + m}}{a_{\alpha}} \prod_{1 \leq s \leq r, \lambda^{(s)} \neq \emptyset} (1 - \mathbf{v}^{-2d_s}) \prod_{i \in I, (m_i, l_i) \neq (0, 0)} (1 - \mathbf{v}^{-2});$$

see (2.5) for the notations.

Proof. By definition, it suffices to show that each equality holds for $q \in \mathfrak{Q}$ with $q \gg 0$. Then (1) follows from [4, Prop. 6.3] and (2) can be proved by an easy calculation.

(3) For each $q \in \mathfrak{Q}$ and $\underline{z} = \underline{z}_{\alpha, q} \in \mathcal{X}_{\mathbb{F}_q}(\underline{d})$,

$$S_q(\alpha, \underline{z}) = S_q(\alpha) \oplus (\oplus_{1 \leq s \leq r} S_q(\lambda^{(s)}, z_s)),$$

so $\{[S_q(\alpha, \underline{z})], \Theta_{\vec{x}, q}\}_q \neq 0$ if and only if $[S_q(\alpha, \underline{z})]$ occurs in the expression of $\Theta_{\vec{x}, q}$ in (2.5). This happens only if $\det \alpha := \det \alpha + \sum_{1 \leq s \leq r} |\lambda^{(s)}| d_s \vec{c} = \vec{x}$ and the length of $\lambda^{(s)}$ is 1 for each $1 \leq s \leq r$. In this case, the coefficient of $[S_q(\alpha, \underline{z})]$ is given by

$$v_q^{l+m} \prod_{1 \leq s \leq r, \lambda^{(s)} \neq \emptyset} (1 - v_q^{-2d_s}) \prod_{i \in I, (m_i, l_i) \neq (0, 0)} (1 - v_q^{-2}).$$

On the other hand,

$$\{[S_q(\alpha, \underline{z})], [S_q(\alpha, \underline{z})]\}_q = \frac{v_q^{(\alpha, \alpha)}}{a_{S_q(\alpha, \underline{z})}}.$$

This completes the proof. \square

Our next goal is to introduce the reduced Drinfeld double of the topological bialgebra $\tilde{\mathcal{H}}_v(\mathbb{X}) = \tilde{\mathcal{H}}_v(\mathbb{X}) \otimes_{\mathbb{Q}(v)} \mathbb{Q}(v)$. By Proposition 5.8, the pairing in (5.4) induces a pairing

$$\{-, -\} : \tilde{\mathcal{H}}_v(\mathbb{X}) \times \tilde{\mathcal{H}}_v(\mathbb{X}) \longrightarrow \mathbb{Q}(v).$$

Consider the pair of $\mathbb{Q}(v)$ -algebras $\tilde{\mathcal{H}}_v^{\pm}(\mathbb{X})$ with bases

$$\{u_{\alpha}^{\pm} Z_{\underline{l}}^{\pm} K_{\mathbf{a}}^{\pm} \mid \alpha \in \mathcal{S}, \underline{l} \in \mathcal{L}, \mathbf{a} \in K_0(\mathbb{X})\}.$$

The *Drinfeld double* of the topological bialgebra $\tilde{\mathcal{H}}_v^{\pm}(\mathbb{X})$ with respect to the pairing $\{-, -\}$ is the associative algebra $\tilde{D}\mathcal{H}_v(\mathbb{X})$, defined as the free product of algebras $\tilde{\mathcal{H}}_v^{+}(\mathbb{X})$ and $\tilde{\mathcal{H}}_v^{-}(\mathbb{X})$ subject to the relations

$$R(a, b) : \sum a_1^{-} b_2^{+} \{a_2, b_1\} = \sum b_1^{+} a_2^{-} \{a_1, b_2\},$$

where $a, b \in \tilde{\mathcal{H}}_v(\mathbb{X})$, $\Delta(a) = \sum a_1 \otimes a_2$, and $\Delta(b) = \sum b_1 \otimes b_2$. The *reduced Drinfeld double* $D\mathcal{H}_v(\mathbb{X})$ is the quotient of $\tilde{D}\mathcal{H}_v(\mathbb{X}) = \tilde{\mathcal{H}}_v^{+}(\mathbb{X}) \otimes \tilde{\mathcal{H}}_v^{-}(\mathbb{X})$ by the Hopf ideal generated by $K_{\mathbf{a}}^{+} \otimes K_{-\mathbf{a}}^{-} - 1 \otimes 1$ for $\mathbf{a} \in K_0(\mathbb{X})$. We then have the following triangular decomposition of $D\mathcal{H}_v(\mathbb{X})$ as $\mathbb{Q}(v)$ -vector spaces

$$\mathcal{H}_v^{+}(\mathbb{X}) \otimes_{\mathbb{Q}(v)} \mathcal{K} \otimes_{\mathbb{Q}(v)} \mathcal{H}_v^{-}(\mathbb{X}) \xrightarrow{\cong} D\mathcal{H}_v(\mathbb{X}),$$

where $\mathcal{K} = \mathbb{Q}(v)[K_0(\mathbb{X})]$. For each $\mathbf{a} \in K_0(\mathbb{X})$, we will write

$$K_{\mathbf{a}} = K_{\mathbf{a}}^{+} \otimes 1 = 1 \otimes K_{\mathbf{a}}^{-} \in D\mathcal{H}_v(\mathbb{X}).$$

Remarks 5.9. (1) Let $\mathcal{C}_v(\mathbb{X})$ denote the $\mathbb{Q}(v)$ -subalgebra of $\mathcal{H}_v(\mathbb{X}) = \mathcal{H}_v(\mathbb{X}) \otimes \mathbb{Q}(v)$ generated by $u_{[\mathcal{O}(\ell\vec{c})]}$ for $\ell \in \mathbb{Z}$, $u_{[S_{i,j}]}$ for $i \in I$, $0 \leq j \leq p_i - 1$, and T_r for $r \geq 1$, called the *generic composition algebra* of \mathbb{X} . It was shown that [28, Th. 5.3] that $\mathcal{C}_v(\mathbb{X})$ is isomorphic to a quantized loop algebra $\mathbf{U}_v(\hat{\mathfrak{n}})$.

(2) In [9], Dou, Jiang and Xiao proved that for any field k with q elements, the double Ringel–Hall algebra of $H(\mathbb{X}_k)$ provides a realization of the quantized loop algebra $\mathbf{U}_{v_q}(\mathcal{L}\mathfrak{g})$ (over \mathbb{C} , specialized at $v = v_q = \sqrt{q}$) of the simple Lie algebra \mathfrak{g} associated with \mathbb{X}_k . Let $D\mathcal{C}_v(\mathbb{X})$ be the subalgebra of $D\mathcal{H}_v(\mathbb{X})$ generated by the set

$$\{u_{[\mathcal{O}(\ell\vec{c})]}^{\pm}, u_{[S_{i,j}]}^{\pm}, T_r^{\pm}, K_{\mathbf{a}}^{\pm} \mid \ell \in \mathbb{Z}, i \in I, 0 \leq j \leq p_i - 1, r \geq 1, \mathbf{a} \in K_0(\mathbb{X})\},$$

called the *generic Drinfeld double composition algebra* of \mathbb{X} . We would expect a generic version of the isomorphism given in [9, Th. 5.5], that is, there is a $\mathbb{Q}(\mathbf{v})$ -algebra isomorphism $\mathbf{U}_{\mathbf{v}}(\mathcal{L}\mathfrak{g}) \rightarrow D\mathcal{C}_{\mathbf{v}}(\mathbb{X})$.

6. HALL POLYNOMIALS FOR TAME QUIVERS

In this section we first define decomposition sequences for a tame quiver Q analogously as in Section 3 and then use Theorem 3.3 together with [6, Prop. 5] to prove the existence of Hall polynomials for each triple of decomposition sequences for Q . This not only refines the main theorem in [14], but also confirms a conjecture in [3, Conj. 3.4].

Throughout this section, $Q = (Q_0, Q_1)$ denotes an acyclic tame quiver, that is, Q contains no oriented cycles and the underlying diagram of Q is an extended Dynkin diagram of type \tilde{A} , \tilde{D} and \tilde{E} . Let kQ be the path algebra of Q over a field k . By $\text{mod } kQ$ we denote the category of finite dimensional (left) kQ -modules. It is well known from [8] that the subcategory $\text{ind } kQ$ of indecomposable kQ -modules admits a disjoint decomposition

$$\text{ind } kQ = \mathcal{P}_k \cup \mathcal{R}_k \cup \mathcal{I}_k,$$

where \mathcal{P}_k (resp., $\mathcal{R}_k, \mathcal{I}_k$) denotes the subcategories of indecomposable preprojective (resp., regular, preinjective) modules. Moreover, \mathcal{R}_k consists of finitely many non-homogeneous tubes and infinitely many homogeneous tubes which are parameterized by a subset \mathbb{H}_k of $\mathbb{P}^1(k)$.

Similar to Section 3, we denote by $\Xi = \Xi(kQ)$ the set of isoclasses of kQ -modules which clearly depends on the ground field k . Let $\Xi_{\mathcal{P}}, \Xi_{\mathcal{R}}$ and $\Xi_{\mathcal{I}}$ be the subsets of Ξ consisting of the isoclasses of preprojective, regular and preinjective modules, respectively. Further, let Ξ_{nh} be the subset of Ξ formed by the isoclasses of kQ -modules without homogeneous regular summands. Hence, the set Ξ_{nh} can be described combinatorially and is independent of k . Moreover, each module whose summands lie in a single homogeneous tube is determined by a partition. For each $\alpha \in \Xi$, we fix a representative $M_k(\alpha)$ in the class α . Given $\alpha, \beta \in \Xi$, we write $\alpha \oplus \beta$ for the isoclass of $M_k(\alpha) \oplus M_k(\beta)$. Thus, each $\alpha \in \Xi$ can be uniquely written as $\alpha = \alpha_{\mathcal{P}} \oplus \alpha_{\mathcal{R}} \oplus \alpha_{\mathcal{I}}$ with $\alpha_{\mathcal{P}} \in \Xi_{\mathcal{P}}, \alpha_{\mathcal{R}} \in \Xi_{\mathcal{R}},$ and $\alpha_{\mathcal{I}} \in \Xi_{\mathcal{I}}$.

A *decomposition sequence of type $\underline{d} = (d_1, \dots, d_r)$* for Q is a pair $\alpha = (\alpha, \lambda)$, where $\alpha \in \Xi_{\text{nh}}$ and λ is a Segre sequence of type \underline{d} (see Section 3 for the definition). For each Segre sequence λ of type \underline{d} and $\underline{z} = (z_1, \dots, z_r) \in \mathcal{X}_k(\underline{d})$, define

$$M_k(\lambda, \underline{z}) := \bigoplus_{i=1}^r M_k(\lambda^{(i)}, z_i) \in \text{mod } kQ,$$

where $M_k(\lambda^{(i)}, z_i)$ is the regular kQ -module in the homogeneous tube associated with the point z_i determined by the partition $\lambda^{(i)}$. Further, for each decomposition sequence $\alpha = (\alpha, \lambda)$ of type \underline{d} , define

$$M_k(\alpha, \underline{z}) := M_k(\alpha) \oplus M_k(\lambda, \underline{z}).$$

Finally, by $\mathcal{M} = \mathcal{M}(Q)$ we denote the set of all decomposition sequences for Q via identifying (α, λ) and (α, μ) , where λ is obtained from μ by inserting and removing some pairs (\emptyset, d) ; see Remark 3.1. Further, we define the subsets $\mathcal{M}_{\mathcal{P}}, \mathcal{M}_{\mathcal{R}}$ and $\mathcal{M}_{\mathcal{I}}$ of \mathcal{M} in a natural sense. Each $\alpha \in \mathcal{M}$ can be written as $\alpha = \alpha_{\mathcal{P}} \oplus \alpha_{\mathcal{R}} \oplus \alpha_{\mathcal{I}}$ with $\alpha_{\mathcal{P}} \in \mathcal{M}_{\mathcal{P}}, \alpha_{\mathcal{R}} \in \mathcal{M}_{\mathcal{R}}$ and $\alpha_{\mathcal{I}} \in \mathcal{M}_{\mathcal{I}}$.

Definition 6.1. Given $\alpha, \beta, \gamma \in \mathcal{M} = \mathcal{M}(Q)$ of type \underline{d} , if there exists a polynomial $\psi_{\alpha, \beta}^\gamma \in \mathbb{Z}[T]$ such that for each finite field k with $q := |k| \gg 0$,

$$\psi_{\alpha, \beta}^\gamma(q) = F_{M_k(\alpha, \underline{z}), M_k(\beta, \underline{z})}^{M_k(\gamma, \underline{z})} \quad \text{for all } \underline{z} \in \mathcal{X}_k(\underline{d}),$$

then the Hall polynomial $\psi_{\alpha, \beta}^\gamma$ is said to exist for α, β and γ .

The main aim of this section is to prove the existence of Hall polynomials for the tame quiver Q . We need some preparation.

Let \mathbb{X}_k be a weighted projective line associated with the tame quiver Q in the sense that there exists an equivalence $D^b(\text{coh-}\mathbb{X}_k) \cong D^b(kQ\text{-mod})$ of the bounded derived categories. More precisely, there exists a tilting bundle T_k in $\text{coh-}\mathbb{X}_k$, whose summands form a complete slice in the Auslander–Reiten quiver of $\text{vect-}\mathbb{X}_k$, such that the endomorphism algebra of T_k is isomorphic to the path algebra kQ . Let $\mathcal{F}(T_k)$ be the torsion-free class of $\text{coh-}\mathbb{X}_k$ induced by T_k which consists of the sheaves $S_k \in \text{coh-}\mathbb{X}_k$ satisfying $\text{Hom}(T_k, S_k) = 0$. Since T_k is a vector bundle, it is easily seen that $\mathcal{F}(T_k)$ is a full subcategory of $\text{vect-}\mathbb{X}_k$, which can be described combinatorially and is independent of the field k . Thus, we will drop the index k and view $\mathcal{F}(T)$ as a subset of \mathcal{S} . Let $\mathcal{T}(T)$ be the subset of \mathcal{S} consisting of $\alpha \in \mathcal{S}$ such that $M_k(\alpha, \underline{z})$ is generated by T_k for each $\underline{z} \in \mathcal{X}_k(\underline{d})$, where \underline{d} is the type of α . Then each decomposition sequence α admits a decomposition $\alpha = \alpha_+ \oplus \alpha_-$ with $\alpha_+ \in \mathcal{T}(T)$ and $\alpha_- \in \mathcal{F}(T)$. Moreover, T induces bijections $\Psi_1 : \mathcal{T}(T) \rightarrow \mathcal{M}_{\mathcal{P} \oplus \mathcal{R}}$ (preserving the Segre sequence) and $\Psi_2 : \mathcal{F}(T) \rightarrow \mathcal{M}_{\mathcal{I}}$. As a consequence, the functor $\text{Hom}(T, -)$ induces a triangulated equivalence of bounded derived categories

$$\mathbf{R}\text{Hom}(T, -) : D^b(\text{coh-}\mathbb{X}_k) \longrightarrow D^b(kQ\text{-mod}).$$

Let $D\mathbf{H}(kQ)$ and $D\mathbf{H}(\mathbb{X}_k)$ denote the double Ringel–Hall algebras of kQ and \mathbb{X}_k over the field $\mathbb{Q}(v_q)$, respectively; see [32, 4, 9]. Applying [6, Prop. 5] gives the following result.

Lemma 6.2. *There exists a $\mathbb{Q}(v_q)$ -algebra isomorphism*

$$D\mathbf{H}(kQ) \xrightarrow{\cong} D\mathbf{H}(\mathbb{X}_k),$$

which takes

$$\begin{aligned} [M_k(\theta_{\mathcal{P}} \oplus \theta_{\mathcal{R}}, \underline{z})]^+ &\longmapsto [S_k(\theta_+, \underline{z})]^+, \\ [M_k(\theta_{\mathcal{I}}, \underline{z})]^+ &\longmapsto v_q^{-(\theta_-, \theta_-)} [S_k(\theta_-, \underline{z})]^- K_{\theta_-}; \end{aligned}$$

where $\theta = \theta_{\mathcal{P}} \oplus \theta_{\mathcal{R}} \oplus \theta_{\mathcal{I}} \in \mathcal{M}$ of type \underline{d} , $\underline{z} \in \mathcal{X}_k(\underline{d})$, $\theta_+ = \Psi_1^{-1}(\theta_{\mathcal{P}} \oplus \theta_{\mathcal{R}})$, and $\theta_- = \Psi_2^{-1}(\theta_{\mathcal{I}})$.

Now let $\mathcal{H}_v(T)$ be the $\mathbb{Q}(v)$ -submodule of $D\mathcal{H}_v(\mathbb{X})$ spanned by the set

$$\{u_{\alpha}^+ Z_{\underline{l}}^+ u_{\beta}^- K_{\mathbf{a}} \mid \alpha \in \mathcal{T}(T), \underline{l} \in \mathcal{L}, \beta \in \mathcal{F}(T), \mathbf{a} \in K_0(\mathbb{X})\}.$$

Proposition 6.3. $\mathcal{H}_v(T)$ is a $\mathbb{Q}(v)$ -subalgebra of $D\mathcal{H}_v(\mathbb{X})$.

Proof. Since the torsion (resp., torsion-free) class induced by T is closed under extension, the $\mathbb{Q}(v)$ -submodule of $D\mathcal{H}_v(\mathbb{X})$ spanned by $u_{\alpha}^+ Z_{\underline{l}}^+$ for $\alpha \in \mathcal{T}(T)$ (resp., by $u_{\beta}^- K_{\mathbf{a}}$ for $\beta \in \mathcal{F}(T)$ and $\mathbf{a} \in K_0(\mathbb{X})$) is a subalgebra of $D\mathcal{H}_v(\mathbb{X})$. Hence, we only need to show that $u_{\beta}^- u_{\alpha}^+ Z_{\underline{l}}^+ \in \mathcal{H}_v(T)$ for $\alpha \in \mathcal{T}(T)$, $\underline{l} \in \mathcal{L}$, and $\beta \in \mathcal{F}(T)$.

Take $\beta \in \mathcal{F}(T)$ and write $\Delta(u_\beta) = \sum b_1 \otimes b_2$. Since $\mathcal{F}(T)$ is closed under subobjects, all the b_2 are generated by u_β with $\beta \in \mathcal{F}(T)$. Then

$$u_\beta^- Z_r^+ = Z_r^+ u_\beta^- + \sum \{Z_r, b_1\} b_2^- \in \mathcal{H}_v(T).$$

It remains to show $u_\beta^- u_\alpha^+ \in \mathcal{H}_v(T)$ for $\alpha \in \mathcal{T}(T)$, $\beta \in \mathcal{F}(T)$. Assume $\Delta(u_\alpha) = \sum a_1 \otimes a_2$. Then the a_1 are generated by u_α , $\alpha \in \mathcal{T}(T)$ and $Z_r, r \geq 1$. By the defining relation $R(u_\alpha, u_\beta)$ in $D\mathcal{H}_v(\mathbb{X})$, we have

$$(6.1) \quad u_\beta^- u_\alpha^+ = \sum \{a_2, b_1\} a_1^+ b_2^-,$$

where $\Delta(u_\beta) = \sum b_1 \otimes b_2$ and $a_1^+ b_2^- \in \mathcal{H}_v(T)$. We need to show that the sum is finite. Since $u_\alpha = u_{\alpha_f} u_{\alpha_t}$, we only need to consider the following two cases:

Case 1: $\alpha \in \mathcal{S}_t$. In this case, a_1 and a_2 are generated by u_α , $\alpha \in \mathcal{T}(T)$. Hence, there are only finitely many choices of a_1 and a_2 since their degrees are bounded by $\deg \alpha$. Furthermore, there are only finitely many b_1 's such that $\{a_2, b_1\} \neq 0$. This forces that there are finitely many choices of b_2 which give nonzero terms in the right hand side of (6.1).

Case 2: $\alpha \in \mathcal{S}_f$. In this case, each term a_2 can be assumed to have the form u_γ for some $\gamma \in \mathcal{S}_f$. Then $\{a_2, b_1\} \neq 0$ implies that $b_1 = c_\gamma u_\gamma$ for some $c_\gamma \in \mathbb{Q}(\mathbf{v})$. Thus, there are an epimorphism $S_k(\beta) \twoheadrightarrow S_k(\gamma)$ and a monomorphism $S_k(\gamma) \hookrightarrow S_k(\alpha)$, which ensures that there are only finitely many choices of γ and so for a_2 . Hence, there are finitely many triples (a_2, b_1, b_2) which contribute a nonzero term in (6.1). \square

Since all the exceptional simple sheaves belong to the torsion classes, we obtain by an argument similar to that in the proof of Proposition 5.4 that the two sets

$$\{u_\alpha^+ T_{\underline{l}}^+ u_\beta^- K_{\mathbf{a}}\} \text{ and } \{u_\alpha^+ \Theta_{\underline{l}}^+ u_\beta^- K_{\mathbf{a}}\},$$

where $\alpha \in \mathcal{T}(T)$, $\underline{l} \in \mathcal{L}$, $\beta \in \mathcal{F}(T)$, and $\mathbf{a} \in K_0(\mathbb{X})$, are both $\mathbb{Q}(\mathbf{v})$ -bases of $\mathcal{H}_v(T)$.

A rational function $\phi(\mathbf{v}) = f(\mathbf{v})/g(\mathbf{v}) \in \mathbb{Q}(\mathbf{v})$ is said to be *nearly integral* if $f(\mathbf{v}), g(\mathbf{v}) \in \mathbb{Z}[\mathbf{v}]$ and $g(\mathbf{v})$ is monic. We say that an element $a \in \mathcal{H}_v(\mathbb{X})$ is generated by a set X with nearly integral coefficients if a is a linear combination of monomials of elements in X whose coefficients are nearly integral functions.

Lemma 6.4. *For each $\theta \in \mathcal{F}(T)$, the element $u_\theta \in \mathcal{H}_v(\mathbb{X})$ can be generated by $\{u_{[\mathcal{O}(\vec{x})]} \mid \vec{x} \in \mathbb{L}\}$ with nearly integral coefficients.*

Proof. By the assumption, $S_k(\theta) = E$ is a vector bundle. If E is decomposable, then E can be written as $E = \bigoplus_{1 \leq i \leq r} E_i^{l_i}$, where E_i are indecomposable satisfying $\text{Hom}(E_i, E_j) = 0$ for $i > j$. Then

$$u_{[E]} = \mathbf{v}^{-\sum_{1 \leq i \leq r} l_i l_{i-1} - \sum_{i < j} l_i l_j \langle E_i, E_j \rangle} \frac{u_{[E_1]}^{l_1}}{[l_1]!} \cdots \frac{u_{[E_r]}^{l_r}}{[l_r]!}.$$

Thus, it suffices to prove the assertion in the case where E is indecomposable.

Now suppose that E is indecomposable. By Lemma 2.1, for each finite field $k = \mathbb{F}_q$, there is an exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0$$

in $\text{vect-}\mathbb{X}_k$ such that L is a line bundle and $\text{Ext}^1(F, L) \cong k$. Then, in the Ringel–Hall algebra $H(\mathbb{X}_k)$, we have the equalities

$$[L][F] = v_q^{\langle L, F \rangle} [L \oplus F] \quad \text{and}$$

$$[F][L] = v_q^{\langle F, L \rangle} (v_q^{2\langle L, F \rangle} [L \oplus F] + \frac{v_q^2 - 1}{a_F} [E]) = v_q^{-1} (v^{\langle L, F \rangle} [L][F] + \frac{v_q^2 - 1}{a_F} [E]).$$

It follows that

$$[E] = \frac{v_q}{v_q^2 - 1} a_F [F][L] - \frac{v_q^{\langle L, F \rangle}}{v_q^2 - 1} a_F [L][F].$$

In other words, we have in $\mathcal{H}_v(\mathbb{X})$,

$$u_{[E]} = \frac{v}{v^2 - 1} a_{\theta'} u_{[F]} u_{[L]} - \frac{v^{\langle L, F \rangle}}{v^2 - 1} a_{\theta'} u_{[L]} u_{[F]},$$

where $\theta' \in \mathcal{F}(T)$ is defined by $S(\theta') = F$. Since F is a vector bundle with rank smaller than that of E , the assertion follows from an induction on the rank of E . \square

Lemma 6.5. *For decomposition sequences $\theta \in \mathcal{F}(T)$ and $\delta \in \mathcal{T}(T)$, the element $u_{\theta}^- u_{\delta}^+$ has nearly integral coefficients with respect to the basis*

$$(6.2) \quad \{u_{\alpha}^+ \Theta_{\underline{l}}^+ u_{\beta}^- K_{\mathbf{a}} \mid \alpha \in \mathcal{T}(T), \underline{l} \in \mathcal{L}, \beta \in \mathcal{F}(T), \mathbf{a} \in K_0(\mathbb{X})\}.$$

Proof. By Lemma 6.4, for each $\gamma \in \mathcal{F}(T)$, u_{γ} can be generated by $\{u_{[\mathcal{O}(\vec{x})]} \mid \vec{x} \in \mathbb{L}\}$ with nearly integral coefficients. Then by Proposition 5.6,

$$(6.3) \quad \Delta(u_{\gamma}) = \sum c_{\gamma} a_{1, \gamma} \otimes a_{2, \gamma}$$

where each $a_{1, \gamma}$ can be generated by $\{u_{[\mathcal{O}(\vec{x})]}, \Theta_{\vec{y}} \mid \vec{x} \in \mathbb{L}, \vec{y} \in \mathbb{L}_+\}$ with nearly integral coefficients, and each $a_{2, \gamma}$ has the form $u_{\gamma'}$ for some $\gamma' \in \mathcal{F}(T)$. Since $u_{[\mathcal{O}(\vec{x})]}$ (resp., $\Theta_{\vec{x}}$) can be generated by $u_{[\mathcal{O}(\vec{l})]}$'s (resp., $\Theta_{r\vec{c}}$'s) and $u_{[S_{i,j}]}$'s with nearly integral coefficients, we conclude that the $a_{1, \gamma}$ in (6.3) can be generated by

$$\{u_{[\mathcal{O}(\vec{l})]}, u_{[S_{i,j}]}, \Theta_{r\vec{c}} \mid l \in \mathbb{Z}, r \in \mathbb{N}, i \in I, 0 \leq j \leq p_i - 1\}$$

with nearly integral coefficients.

We first consider the following two special cases of δ .

Case 1: $\delta \in \mathcal{S}_t$. In this case, $\Delta(u_{\delta}) = \sum c_{\delta} u_{\delta_1} \otimes u_{\delta_2}$, where $\delta_1, \delta_2 \in \mathcal{S}_t$ and $c_{\delta} = v^{\langle u_{\delta_1}, u_{\delta_2} \rangle} F_{\delta_1, \delta_2}^{\delta}$ are nearly integral functions. Hence,

$$u_{\theta}^- u_{\delta}^+ = \sum c_{\theta} c_{\delta} \{a_{1, \theta}, u_{\delta_2}\} u_{\delta_1}^+ a_{2, \theta}^-,$$

where the coefficients $c_{\theta} c_{\delta} \{a_{1, \theta}, u_{\delta_2}\}$ are nearly integral functions by Proposition 5.8(3).

Case 2: $\delta \in \mathcal{S}_f$. In this case, $u_{\theta}^- u_{\delta}^+ = \sum c_{\theta} c_{\delta} \{a_{1, \theta}, a_{2, \delta}\} a_{1, \delta}^+ a_{2, \theta}^-$. Note that $a_{2, \delta}$ has the form $u_{\delta'}$ for some $\delta' \in \mathcal{S}_f$. Thus, $\{a_{1, \theta}, a_{2, \delta}\} \neq 0$ if and only if $u_{\delta'}$ is a nonzero term in the linear combination of $a_{1, \theta}$ with respect to the third basis in Proposition 5.4. By Proposition 5.8, the coefficients $c_{\gamma} c_{\delta} \{a_{1, \theta}, a_{2, \delta}\}$ are nearly integral functions.

In general, u_{δ} has a decomposition $u_{\delta} = u_{\delta_f} u_{\delta_t}$, where $\delta_f \in \mathcal{S}_f$ and $\delta_t \in \mathcal{S}_t$. Hence,

$$u_{\theta}^- u_{\delta}^+ = u_{\theta}^- u_{\delta_f}^+ u_{\delta_t}^+ = \sum c_{\theta} c_{\delta_f} \{a_{1, \theta}, a_{2, \delta_f}\} a_{1, \delta_f}^+ a_{2, \theta}^- u_{\delta_t}^+$$

$$= \sum c_{\theta} c_{\delta_f} c_{\theta'} c_{\delta_t} \{a_{1, \theta}, a_{2, \delta_f}\} \{a_{1, \theta'}, u_{\delta_{t,2}}\} a_{1, \delta_f}^+ u_{\delta_{t,1}}^+ a_{2, \theta'}^-,$$

where each $a_{2, \theta}$ takes the form $u_{\theta'}$ for some $\theta' \in \mathcal{F}(T)$, and the coefficients are nearly integral.

Finally, by the construction of T_r given in [4, Sect. 6], we obtain that for each $i \in I$, $[u_{[S_{i,j}], T_r}]$ can be generated by $u_{[S_{i,s}]}, 0 \leq s \leq p_i - 1$, with nearly integral coefficients. An analogous result holds for $[u_{S_{i,j}}, \Theta_{r\bar{c}}]$ by using (5.2). This together with Proposition 5.5 (5) completes the proof. \square

Theorem 6.6. *For arbitrary $\alpha, \beta, \gamma \in \mathcal{M}$ of type \underline{d} , the Hall polynomial $\psi_{\alpha, \beta}^\gamma$ exists.*

Proof. For each $\theta \in \mathcal{M}$ of type \underline{d} , write $\theta = \theta_{\mathcal{P}} \oplus \theta_{\mathcal{R}} \oplus \theta_{\mathcal{I}}$. Further, set

$$\theta_+ = \Psi_1^{-1}(\theta_{\mathcal{P}} \oplus \theta_{\mathcal{R}}) \in \mathcal{T}(T) \quad \text{and} \quad \theta_- = \Psi_2^{-1}(\theta_{\mathcal{I}}) \in \mathcal{F}(T).$$

Combining with Proposition 5.5 and Lemma 6.5, we obtain that in the expression of the product $(u_{\alpha_+}^+ u_{\alpha_-}^-)(u_{\beta_+}^+ u_{\beta_-}^-)$ with respect of the basis (6.2) of $\mathcal{H}_v(T)$, the coefficient of $u_{\gamma_+}^+ u_{\gamma_-}^- K_{\alpha_- + \beta_- - \gamma_-}$ is a nearly integral function $\xi_{\alpha, \beta}^\gamma(v)$. For each finite field k with $q = |k| \gg 0$ and $\underline{z} \in \mathcal{X}_k(\underline{d})$, we take a total ordering \preccurlyeq in \mathbb{H}_k so that $\underline{z} = \underline{z}_{\alpha, q} = \underline{z}_{\beta, q} = \underline{z}_{\gamma, q}$ in defining the embedding Φ in Proposition 5.2. Then $\xi_{\alpha, \beta}^\gamma(v_q)$ is the coefficient of the basis element $[S_k(\gamma_+, \underline{z})]^+ [S_k(\gamma_-, \underline{z})]^- K_{\alpha_- + \beta_- - \gamma_-}$ in the expression of the product

$$([S_k(\alpha_+, \underline{z})]^+ [S_k(\alpha_-, \underline{z})]^-) ([S_k(\beta_+, \underline{z})]^+ [S_k(\beta_-, \underline{z})]^-).$$

By Lemma 6.2, there exists an integer $l(\alpha, \beta, \gamma)$, depending on α, β and γ , such that $v_q^{l(\alpha, \beta, \gamma)} \xi_{\alpha, \beta}^\gamma(v_q)$ is the coefficient of the basis element $[M_k(\gamma, \underline{z})]^+$ in the product $[M_k(\alpha, \underline{z})]^+ [M_k(\beta, \underline{z})]^+$. Set $\eta_{\alpha, \beta}^\gamma(v) = v^{l(\alpha, \beta, \gamma)} \xi_{\alpha, \beta}^\gamma(v)$, which is again a nearly integral function. Then, by the definition of the multiplication in the Ringel–Hall algebra $H(kQ)$ of kQ , $\eta_{\alpha, \beta}^\gamma(v)$ takes integer values at $v_q = \sqrt{q}$ for prime powers $q \gg 0$. This forces that $\eta_{\alpha, \beta}^\gamma(v)$ is an integer polynomial in v^2 , that is, there is a polynomial $\psi_{\alpha, \beta}^\gamma(T) \in \mathbb{Z}[T]$ such that $\psi_{\alpha, \beta}^\gamma(v^2) = \eta_{\alpha, \beta}^\gamma(v)$, as desired. \square

We now consider a special case of the theorem above. Let $\alpha, \beta \in \mathcal{M}$ be such that $M_k(\alpha) = I$ and $M_k(\beta) = P$ are preinjective and preprojective kQ -modules, respectively, where k is a finite field. Further, let $d \geq 1$ and λ be a Segre sequence of type (d) (i.e., λ is a partition). Set $\gamma = (0, \lambda) \in \mathcal{M}$. Then $\chi_k(d)$ consists of points in \mathbb{H}_k of degree d and for each $z \in \chi_k(d)$, $\mathcal{I}_\lambda(z) := M_k(\gamma, z)$ is a regular kQ -module whose summands all lie in the homogeneous tube corresponding to z . Applying Theorem 6.6 gives the Hall polynomial $\psi_{\alpha, \beta}^\gamma$. Therefore, for each finite field k and $z_1, z_2 \in \chi_k(d)$,

$$(6.4) \quad F_{I, P}^{\mathcal{I}_\lambda(z_1)} = F_{I, P}^{\mathcal{I}_\lambda(z_2)}.$$

Following the notation in [3, 2.5], for three kQ -modules A, B, C , set

$$F_C^{A, B} = \frac{|\text{Ext}_{kQ}^1(B, A)_C|}{|\text{Hom}_{kQ}(B, A)|}.$$

Since $|\text{Aut}(\mathcal{I}_\lambda(z_1))| = |\text{Aut}(\mathcal{I}_\lambda(z_2))|$ and $\text{Hom}_{kQ}(I, P) = 0$, applying Lemma 2.2 to (6.4) gives the equality

$$F_{\mathcal{I}_\lambda(z_1)}^{P, I} = |\text{Ext}_{kQ}^1(I, P)_{\mathcal{I}_\lambda(z_1)}| = |\text{Ext}_{kQ}^1(I, P)_{\mathcal{I}_\lambda(z_2)}| = F_{\mathcal{I}_\lambda(z_2)}^{P, I}$$

for all $z_1, z_2 \in \chi_k(d)$. This is exactly the equality conjectured in [3, Conj. 3.4].

Corollary 6.7. *Conjecture 3.4 in [3] holds.*

The following result is a consequence of Theorem 6.6 whose proof is analogous to that of Corollary 4.1.

Corollary 6.8. *Let k be a finite field and fix three kQ -modules M, N, Z . Then there exists a polynomial $\psi_{M,N}^Z \in \mathbb{Z}[T]$ such that for each conservative field extension K of k relative to $\{M, N, Z\}$,*

$$\psi_{M,N}^Z(|K|) = F_{M^K, N^K}^{Z^K}.$$

Acknowledgement. We would like to thank Jie Xiao for stimulating discussions and helpful comments. Especially, the idea of studying Hall polynomials for tame quivers via weighted projective lines comes from his suggestion.

REFERENCES

- [1] P. Baumann and C. Kassel, *The Hall algebra of the category of coherent sheaves on the projective line*, J. reine angew. Math. (2001), 207–233.
- [2] K. Bongratz and D. Dudek, *Decomposition classes for representations of tame quivers*, J. Algebra **240** (2001), 268–288.
- [3] A. Berenstein and J. Greenstein, *Primitively generated Hall algebras*, arXiv:1209.2770.
- [4] I. Burban and O. Schiffmann, *The composition Hall algebra of a weighted projective line*, J. reine angew. Math. **679** (2013), 75–124.
- [5] X. W. Chen and H. Krause, *Introduction to coherent sheaves over weighted projective lines*, arXiv: 0911.4473.
- [6] T. Cramer, *Double Hall algebras and derived equivalences*, Adv. in Math. **224** (2010), 1097–1120.
- [7] B. Deng, J. Du, B. Parshall and J. Wang, *Finite dimensional algebras and quantum groups*, Mathematical Surveys and Monographs Volume 150, Amer. Math. Soc., Providence 2008.
- [8] V. Dlab and C. M. Ringel, *Indecomposable Representations of Graphs and Algebras*, Memoirs Amer. Math. Soc., no. 173, Amer. Math. Soc., Providence, 1976.
- [9] R. Dou, Y. Jiang and J. Xiao, *The Hall algebra approach to Drinfeld’s presentation of quantum loop algebras*, Adv. Math. **231** (2012), 2593–2625.
- [10] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, Singularities, representation of algebras, and vector bundles, Springer Berlin Heidelberg. (1987), 265–297.
- [11] J. A. Green, *Hall algebras, hereditary algebras and quantum groups*, Invent. Math. **120** (1995), 361–377.
- [12] J. Y. Guo, *The Hall polynomials of a cyclic serial algebra*, Comm. Algebra **23** (1995), 743–751.
- [13] P. Hall, *The algebra of partitions*, in: *Proceedings of the 4th Canadian Mathematical Congress, Banff 1957*, University of Toronto Press, Toronto, 1959, pp. 147–159.
- [14] A. Hubery, *Hall polynomials for affine quivers*, Representation Theory **14**(10) (2010), 355–378.
- [15] A. Hubery, *Ringel-Hall algebras of cyclic quivers*, São Paulo J. Math. Sci. **4**(3) (2010), 351–398.
- [16] G. Lusztig, *Quivers, perverse sheaves, and the quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 366–421.
- [17] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd. edition, Oxford Math. Monographs (The Clarendon Press, Oxford Univ. Press, New York, 1995).
- [18] L. Peng, *Some Hall polynomials for representation-finite trivial extension algebras*, J. Algebra **197** (1997), 1–13.
- [19] M. Reineke, *Counting rational points of quiver moduli*, Internat. Math. Res. Notices (2006).
- [20] C. Riedtmann, *Lie algebras generated by indecomposables*, J. Algebra **170**(2) (1994), 526–546.
- [21] C. M. Ringel, *Hall algebras*, in: *Topics in Algebra, Part 1*, S. Balcerzyk et al. (eds.), Banach Center Publications, no. 26, 1990, pp. 433–447.
- [22] C. M. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), 583–591.
- [23] C. M. Ringel, *Hall polynomials for the representation-finite hereditary algebras*, Adv. Math. **84** (1990), 137–178.
- [24] C. M. Ringel, *Hall algebras*, in: *Topics in Algebra, Part 1*, S. Balcerzyk et al. (eds.), Banach Center Publications, no. 26, 1988, pp. 433–447.

- [25] C. M. Ringel, *The composition algebra of a cyclic quiver. Towards an explicit description of the quantum group of type \tilde{A}_n* , Proc. London Math. Soc. (3) **66** (1993), 507–537.
- [26] C. M. Ringel, *Green’s theorem on Hall algebras*, in: *Representation Theory of Algebras and Related Topics*, R. Bautista, R. Martínez-Villa, & J. Peña (eds.), Can. Math. Soc. Conf. Proceedings, no. 19, Amer. Math. Soc., Providence, 1996, pp. 185–245.
- [27] O. Schiffmann, *The Hall algebra of a cyclic quiver and canonical bases of Fock spaces*, Internat. Math. Res. Notices (2000), 413–440.
- [28] O. Schiffmann, *Noncommutative projective curves and quantum loop algebras*, Duke Math. J. **121** (2004), 113–168.
- [29] O. Schiffmann, *Lectures on Hall algebras*, Geometric methods in representation theory. II, 1–141, Sémin. Congr., 24-II, Soc. Math. France, Paris, 2012.
- [30] B. Sevenhant and M. Van den Bergh, *A relation between a conjecture of Kac and the structure of the Hall algebra*, J. Pure Appl. Algebra **160** (2001), 319–332.
- [31] E. Steinitz, *Zur Theorie der Abel’schen Gruppen*, Jahrsber. Deutsch. Math-Verein. **9**(1901), 80–85.
- [32] J. Xiao, *Drinfeld double and Ringel–Green theory of Hall algebras*, J. Algebra **190** (1997), 100–144.

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